Eigenvalue densities of the Wilson Dirac operator in the infrared limit and RMT

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Trento, October 25th, 2011

Outline

- Introduction of the Model
- The Joint Probability Densities
- The Eigenvalue Densities
- Conclusions and Outlook

Introduction of the Model

Partition function in ϵ-regime for N_f flavors

$$Z \propto \int_{\mathrm{U}(N_\mathrm{f})} \exp[\mathcal{L}(U)] \det{}^{\nu} U d\mu(U)$$

Lagrangian of the Goldstone bosons:

$$\begin{aligned}\mathcal{L}(U) &= \frac{\Sigma V}{2} \mathrm{tr}\,(\widetilde{M}_\mathrm{R} U + U^\dagger \widetilde{M}_\mathrm{L}) \\ &\quad - V W_6 \widetilde{a}^2 [\mathrm{tr}\,(U + U^\dagger)]^2 - V W_7 \widetilde{a}^2 [\mathrm{tr}\,(U - U^\dagger)]^2 \\ &\quad - V W_8 \widetilde{a}^2 \mathrm{tr}\,(U^2 + {U^\dagger}^2)\end{aligned}$$

- index of the Dirac operator: ν
- masses for right- and left-handed particles: $\widetilde{M}_{\mathrm{R/L}} = \widetilde{M} \pm \widetilde{\Lambda}$
- lattice spacing: $\widetilde{a}$
- low energy constants: Σ, W_6, W_7, W_8
- spacetime volume: V

a^2-terms of the potential:

$$VW_6\widetilde{a}^2[\mathrm{tr}\,(U+U^\dagger)]^2 + VW_7\widetilde{a}^2[\mathrm{tr}\,(U-U^\dagger)]^2 + VW_8\widetilde{a}^2\mathrm{tr}\,(U^2+{U^\dagger}^2)$$

For $\mathrm{SU}(2)$:

- p-regime: Sharpe and Singleton (1998)
- ϵ-regime: Bär, Necco and Schaefer (2009)

For general number of flavors:

- p-regime: Bär, Rupak and Shoresh (2004)
- p-regime: Sharpe (2006)

Simplification

$$\begin{aligned} & \exp\left[VW_6\tilde{a}^2[\mathrm{tr}\,(U+U^\dagger)]^2 + VW_7\tilde{a}^2[\mathrm{tr}\,(U-U^\dagger)]^2\right] \\ = \; & \frac{1}{2\pi}\int d[m_6,\lambda_7]\exp\left[-\frac{{m_6}^2+{\lambda_7}^2}{2}\right] \\ \times \; & \exp\left[\sqrt{VW_6}\tilde{a}m_6\mathrm{tr}\,(U+U^\dagger) + \sqrt{VW_7}\tilde{a}\lambda_7\mathrm{tr}\,(U-U^\dagger)\right] \end{aligned}$$

- m_6, λ_7 can be considered as additional masses

$\Rightarrow$ omitting the squared trace terms

- can be introduced later on

Random Matrix Ensemble

$$D_{\mathrm{W}} = \begin{pmatrix} aA & W \\ -W^\dagger & aB \end{pmatrix} + m\mathbb{1}_{2n+\nu}$$

distributed by

$$P(D_{\mathrm{W}}) \propto \exp\left[-\frac{n}{2}(\mathrm{tr}\,A^2 + \mathrm{tr}\,B^2) - n\,\mathrm{tr}\,WW^\dagger\right]$$

- Hermitian matrices A ($n \times n$) and B ($(n+\nu) \times (n+\nu)$) are the Wilson-terms $\Rightarrow$ breaking of chiral symmetry
- complex W ($n \times (n+\nu)$) matrix
- at $a = 0$: chGUE describing continuum QCD (Shuryak, Verbaarschot; 1993)
- corresponds to $W_8 > 0$

Damgaard, Splittorff, Verbaarschot (2010)

Microscopic Limit

Partition function for N_{f} flavors

$$Z \propto \int d[D_{\mathrm{W}}]P(D_{\mathrm{W}})\prod_{j=1}^{N_{\mathrm{f}}}\det(D_{\mathrm{W}}(m_j) - \lambda_j\gamma_5)$$

- λ: eigenvalues for $D_5 = D_{\mathrm{W}}\gamma_5$ with $\gamma_5 = \mathrm{diag}\,(\mathbb{1}_n, -\mathbb{1}_{n+\nu})$
- spacetime volume V / matrix dimension $n \to \infty$
- fixed parameters:
  - $\Sigma V\,\mathrm{diag}\,(\widetilde{M}_{\mathrm{R}}, \widetilde{M}_{\mathrm{L}}) = 2n\,\mathrm{diag}\,(m+\lambda, m-\lambda) = \mathrm{diag}\,(\widehat{M}_{\mathrm{R}}, \widehat{M}_{\mathrm{L}})$
  - $\sqrt{VW_8}\widetilde{a} = \sqrt{n/2}a = \widehat{a}$

Outcome

$$Z \propto \int_{\mathrm{U}(N_{\mathrm{f}})} \exp[\mathcal{L}(U)] \det{}^{\nu} U d\mu(U)$$

Lagrangian of the Goldstone bosons:

$$\mathcal{L}(U) \;=\; \mathrm{tr}\,(\widehat{M}_{\mathrm{R}} U + U^{\dagger} \widehat{M}_{\mathrm{L}}) - \widehat{a}^2 \mathrm{tr}\,(U^2 + {U^{\dagger}}^2)$$

Damgaard, Splittorff, Verbaarschot (2010)

The Joint Probability Densities

Properties of D_W and D_5

- D_W is γ_5-Hermitian $\Leftrightarrow$ $D_5 = D_\mathrm{W}\gamma_5$ is Hermitian
- form invariance:
  $D_5:\ V \in \mathrm{U}(2n+\nu),\ V^{-1} = V^\dagger$, compact
  $D_\mathrm{W}:\ U \in \mathrm{U}(n, n+\nu),\ U^{-1} = \gamma_5 U^\dagger \gamma_5$, non-compact
- diagonalization:

$$
\begin{aligned}
D_5 &= VxV^{-1} \\
D_\mathrm{W} &= U \left(\begin{array}{cc|cc} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ \hline 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{array}\right) U^{-1}
\end{aligned}
$$

x, x_j, y_2 are real diagonal

Diagonalization of $D_{\rm W}$ and D_5

- $$D_5 = VxV^{-1}$$

  x: $2n+\nu$ dim $\Rightarrow$ pure real spectrum

- Let $0 \leq l \leq n$

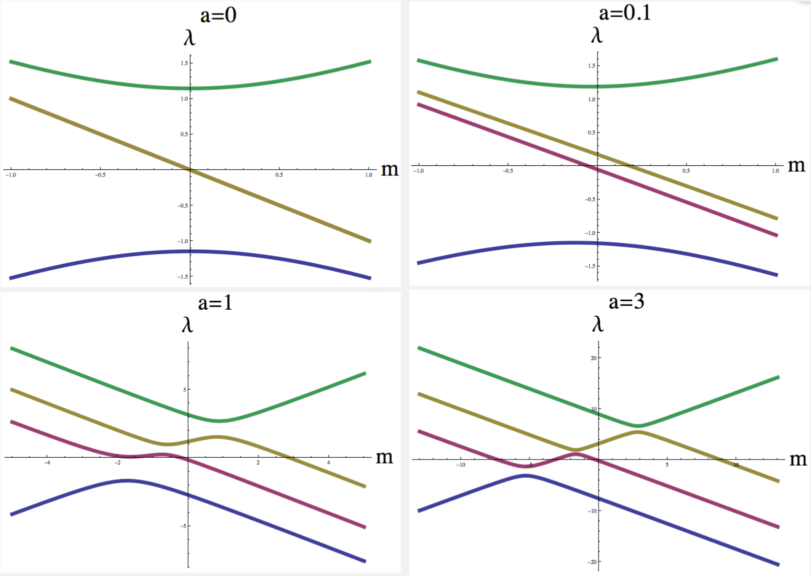
  $$D_{\rm W} \quad = \quad U \left(\begin{array}{cc|cc} x_1 & 0 & 0 & 0 \\ 0 & x_2 & y_2 & 0 \\ \hline 0 & -y_2 & x_2 & 0 \\ 0 & 0 & 0 & x_3 \end{array}\right) U^{-1}$$

  x_1: l dim $\Rightarrow$ real spectrum

  x_2, y_2: $n - l$ dim $\Rightarrow$ complex conj. eigenvalue pairs $x_2 \pm \imath y_2$

  x_3: $\nu + l$ dim $\Rightarrow$ real spectrum

Spectral flow of D_5 (Example for $n = 1$ and $\nu = 2$)



Definition of the Joint Probability Density

Let f be arbitrary integrable function invariant under $\mathrm{Gl}(2n+\nu,\mathbb{C})$:

$$\begin{aligned}\int f(D_5)P(D_{\rm W})d[D_{\rm W}] &= \int f(x)p_5(x)d[x] \\ \int f(D_{\rm W})P(D_{\rm W})d[D_{\rm W}] &= \sum_{l=0}^{n}\int f(z^{(l)})p_{\rm W}^{(l)}(z^{(l)})d[z^{(l)}] \\ &= \int f(z)p_{\rm W}(z)d[z]\end{aligned}$$

where

$$\begin{aligned}p_{\rm W}(z) &= \sum_{l=0}^{n}\int p_{\rm W}^{(l)}(z^{(l)})\delta\left(z-z^{(l)}\right)d[z^{(l)}], \\ z^{(l)} &= \mathrm{diag}\left(x_1, x_2+\imath y_2, x_2-\imath y_2, x_3\right)\end{aligned}$$

Results

For D_5 a $2(n+\nu)$ dim Pfaffian:

$$p_5(x) \propto \Delta_{2n+\nu}(x)\mathrm{Pf}\begin{bmatrix} g_2(x_a, x_b) & x_a^{b-1} g_1(x_a) \\ -x_b^{a-1} g_1(x_b) & 0 \end{bmatrix}$$

degenerated quark mass m

Akemann, Nagao (2011)

For D_W a $n+\nu$ dim determinant:

$$p_\mathrm{W}(z) \propto \Delta_{2n+\nu}(z)$$
$$\times \quad \det\begin{bmatrix} g_\mathrm{c}(z_{a\mathrm{R}})\delta^{(2)}(z_{a\mathrm{R}} - z_{b\mathrm{L}}^*) + g_\mathrm{r}(x_{a\mathrm{R}}, x_{b\mathrm{L}})\delta(y_{a\mathrm{R}})\delta(y_{b\mathrm{L}}) \\ x_{b\mathrm{L}}^{a-1} g_1(x_{b\mathrm{L}})\delta(y_{b\mathrm{L}}) \end{bmatrix}$$

degenerated source term $\lambda\gamma_5$

Kieburg, Verbaarschot, Zafeiropoulos (2011)

Remark: Vandermonde determinant $\Delta_{2n+\nu}(x) = \prod_{i<j}(x_i - x_j)$

The Eigenvalue Densities

Definition

For D_5

$$\rho_5(x_1) = \int p_5(x) d[x_{\neq 1}]$$

$\Rightarrow$ one level density ρ_5

For D_{W}

$$\begin{aligned}
\rho_{\mathrm{R}}(x_{1\mathrm{R}})\delta(y_{1\mathrm{R}}) + \frac{1}{2}\rho_{\mathrm{c}}(z_{1\mathrm{R}}) &= \int p_{\mathrm{W}}(z) d[z_{\neq 1\mathrm{R}}] \\
\rho_{\mathrm{L}}(x_{1\mathrm{L}})\delta(y_{1\mathrm{L}}) + \frac{1}{2}\rho_{\mathrm{c}}(z_{1\mathrm{L}}) &= \int p_{\mathrm{W}}(z) d[z_{\neq 1\mathrm{L}}]
\end{aligned}$$

$\Rightarrow$ three level densities ρ_{c} and

$$\begin{aligned}
\rho_{\mathrm{r}} &= 2\rho_{\mathrm{R}} = \rho_{\mathrm{R}} + \rho_{\mathrm{L}} - \rho_{\chi} \\
\rho_{\chi} &= \rho_{\mathrm{R}} - \rho_{\mathrm{L}}
\end{aligned}$$

In the Microscopic Limit

For $\widehat{a} = 0$:

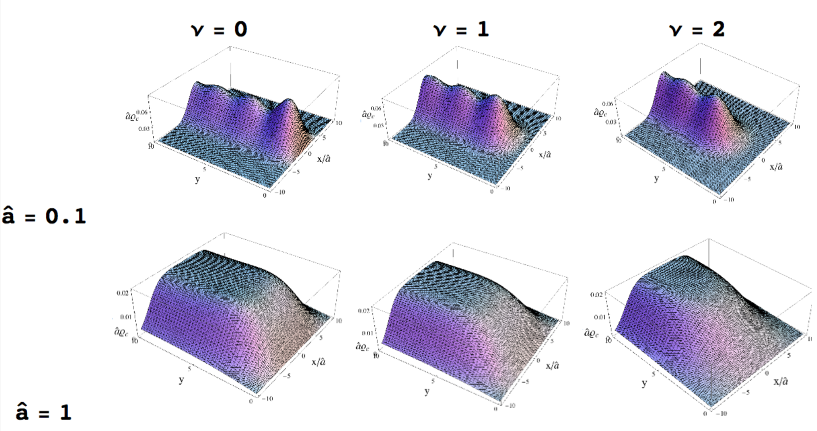
$$\begin{aligned} \rho(y) &= \nu\delta(y) + \frac{y}{2}(J_\nu^2(y) - J_{\nu-1}(y)J_{\nu-2}(y)) \\ &= \nu\delta(y) + \rho_{\rm i}^{(\nu)}(y) \end{aligned}$$

For $\widehat{a} \neq 0$:
two-fold integrals for $\rho_5, \rho_{\rm c}, \rho_{\rm r}, \rho_\chi$

Akemann, Damgaard, Kieburg, Nagao, Splittorff, Verbaarschot, Zafeiropoulos (2010/11)

The Density ρ_{c}

$$\rho_{\mathrm{c}}(z) = \begin{cases} \dfrac{\exp[-x^2/8\hat{a}^2]}{\sqrt{8\pi}\hat{a}} \dfrac{|y|}{|z|} \rho_{\mathrm{i}}^{(\nu)}(|z|), & \hat{a} \ll 1 \\ \dfrac{\Theta(8\hat{a}^2 - |x|)}{16\pi\hat{a}^2} \mathrm{erf}\left[\dfrac{|y|}{\sqrt{8}\hat{a}}\right], & \hat{a} \gg 1 \end{cases}$$

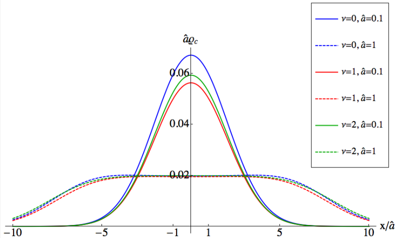


Kieburg, Verbaarschot, Zafeiropoulos (2011)

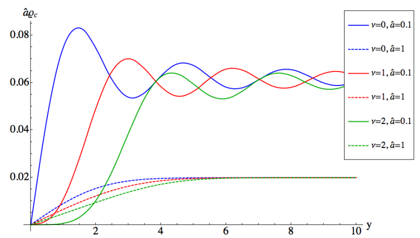
The Density ρ_c

$$\rho_c(z) = \begin{cases} \dfrac{\exp[-x^2/8\hat{a}^2]}{\sqrt{8\pi}\hat{a}} \dfrac{|y|}{|z|} \rho_i^{(\nu)}(|z|), & \hat{a} \ll 1 \\ \dfrac{\Theta(8\hat{a}^2 - |x|)}{16\pi\hat{a}^2} \mathrm{erf}\left[\dfrac{|y|}{\sqrt{8}\hat{a}}\right], & \hat{a} \gg 1 \end{cases}$$

along $y = 5\hat{a}$ axis



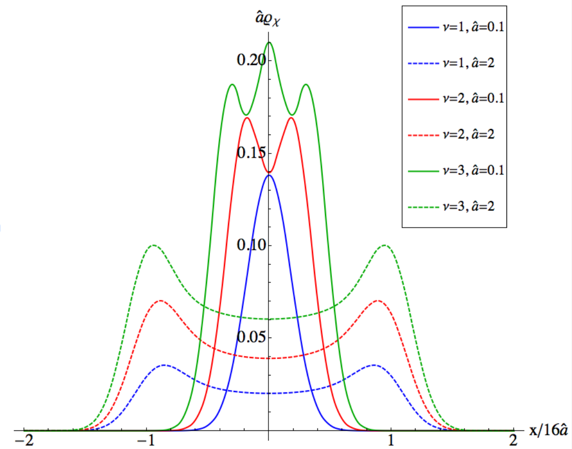
along imaginary axis



Kieburg, Verbaarschot, Zafeiropoulos (2011)

The Density ρ_χ

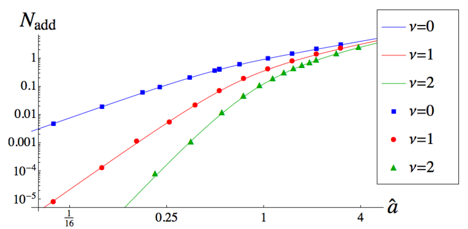
$$\rho_\chi(x) = \begin{cases} \rho^{(\nu)}_{\mathrm{GUE}}\left(\dfrac{x}{4\hat{a}}\right), & \hat{a} \ll 1 \\ \dfrac{\nu\Theta(8\hat{a}^2 - |x|)}{\pi\sqrt{64\hat{a}^4 - x^2}}, & \hat{a} \gg 1 \end{cases}$$



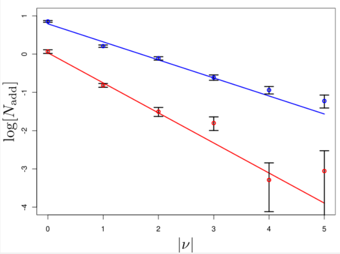
Akemann, Damgaard, Splittorff, Verbaarschot (2010/11)

Number of additional real modes $N_{\rm add}$

$$
\begin{aligned}
N_{\rm add} &= \int \rho_{\rm r}(x)\,dx \\
&= \begin{cases} \dfrac{[2(\nu+1)]!}{[(\nu+1)!]^2}\left(\dfrac{\hat{a}^2}{2}\right)^{\nu+1}, & \hat{a} \ll 1 \\ \left(\dfrac{2}{\pi}\right)^{3/2}\hat{a}, & \hat{a} \gg 1 \end{cases}
\end{aligned}
$$



Kieburg, Verbaarschot, Zafeiropoulos (2011)



Deuzeman, Wenger, Wuilloud (2011)

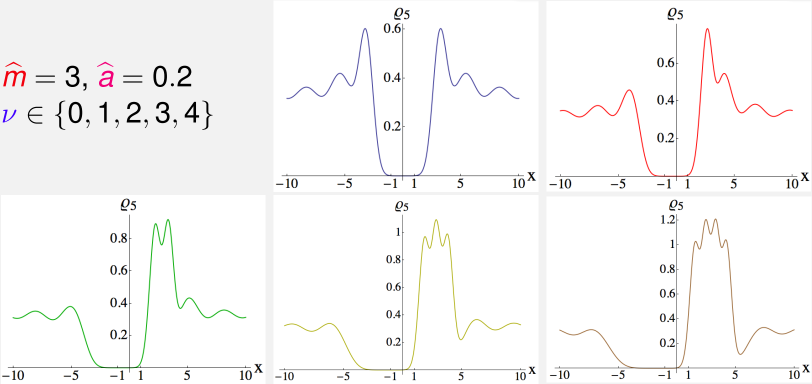
ρ_5 at small Lattice Spacing

$$\rho_5(x) = \rho_{\rm GUE}^{(\nu)}\left(\frac{x-\widehat{m}}{4\widehat{a}}\right) + \frac{|x|}{\sqrt{x^2-\widehat{m}^2}}\rho_{\rm i}^{(\nu)}(\sqrt{x^2-\widehat{m}^2})$$

$$= \rho_\chi(x-\widehat{m}) + \sqrt{8\pi}\widehat{a}\exp\left[-\frac{\widehat{m}^2}{8\widehat{a}^2}\right]\rho_{\rm c}({\rm Re} = \imath\widehat{m}, {\rm Im} = x)$$

$\widehat{m} = 3$, $\widehat{a} = 0.2$

$\nu \in \{0, 1, 2, 3, 4\}$



Akemann, Damgaard, Splittorff, Verbaarschot (2010/11)

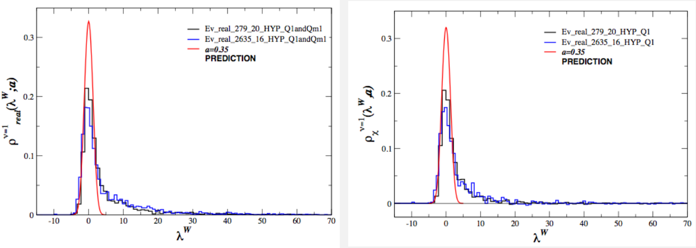
Conclusions and Outlook

Summary

- joint probability densities of D_{W} and D_5
- small $\widehat{a} \ll 1$:
  - broadening of ν formerly zero modes by GUE
  - Gaussian broadening of $\rho_{\mathrm{i}}(y)$ for D_{W}
  - additional real modes are strongly suppressed with increasing ν
- large $\widehat{a} \gg 1$:
  - finite support of size $\approx \widehat{a}^2$ along and parrallel to the real axis for D_{W}
  - ν independence of D_{W} and D_5

Outlook

- unquenched case for non-degenerated masses
- higher correlation functions
- comparison with lattice data



Damgaard, Heller, Splittorff (2011)

- single eigenvalue distributions

To be continued. . .

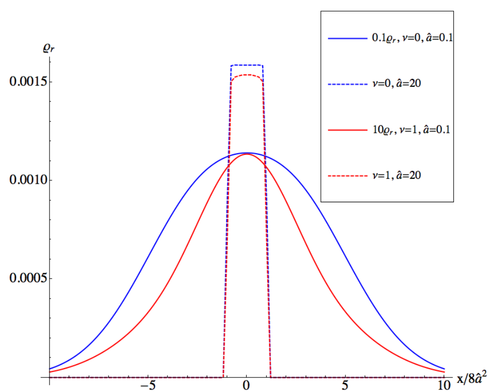
Thank you for your attention!

Collaborators:
Gernot Akemann
Poul H. Damgaard
Kim Splittorff
Jacobus J. M. Verbaarschot
Savvas Zafeiropoulos

Appendix

The Density $\rho_{\rm r}$

$$\rho_{\rm r}(x) = \begin{cases} \hat{a}^{2\nu+1}\int_1^{\sqrt{2}} p_\nu\left(\frac{x^2}{\hat{a}^2}, \lambda^2-1\right)\exp\left[-\frac{x^2}{16\hat{a}^2}\lambda^2\right]d\lambda, & \hat{a} \ll 1 \\ \frac{\Theta(8\hat{a}^2-|x|)}{(2\pi)^{3/2}2\hat{a}}, & \hat{a} \gg 1 \end{cases}$$

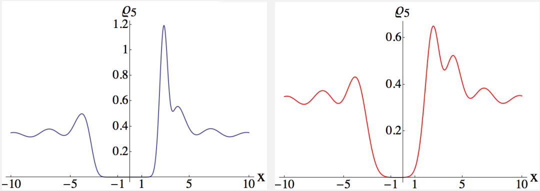


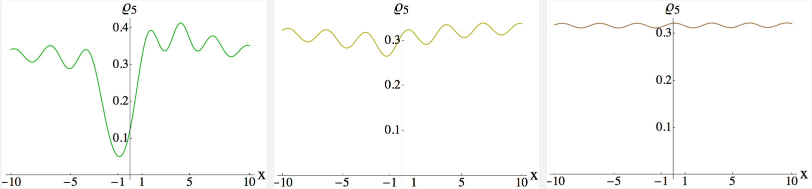
Kieburg, Verbaarschot, Zafeiropoulos (2011)

Lattice Spacing Dependence of ρ_5

$$\rho_5(x) = \begin{cases} \rho_{\mathrm{GUE}}^{(\nu)}\left(\dfrac{x-\hat{m}}{4\hat{a}}\right) + \dfrac{|x|}{\sqrt{x^2-\hat{m}^2}}\rho_{\mathrm{i}}^{(\nu)}(\sqrt{x^2-\hat{m}^2}), & \hat{a} \ll 1 \\ \dfrac{1}{\pi}\left[1+\dfrac{\cos^2 x}{8\hat{a}^2}\right], & \hat{a} \gg 1 \end{cases}$$

$\hat{m} = 3,\ \nu = 1$
$\hat{a} \in \left\{\frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, 2\right\}$





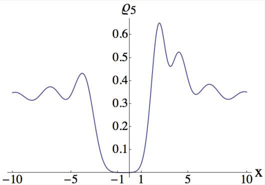
Mean Field Limit of ρ_5

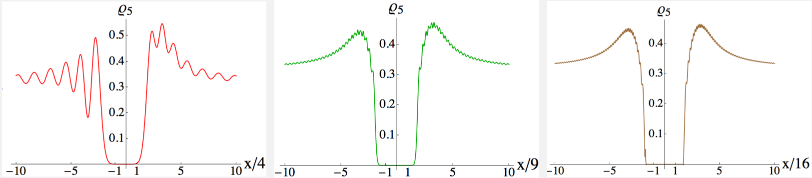
Scaling:

- $\widehat{m} \to s^2 \widehat{m}$
- $x \to s^2 x$
- $\widehat{a} \to s\widehat{a}$

$\widehat{m} = 3$, $\nu = 1$, $\widehat{a} = 0.25$

$s \in \{1, 2, 3, 4\}$



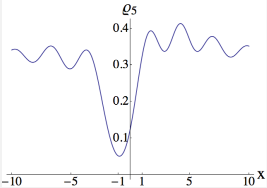


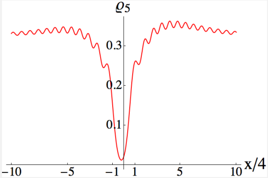
Mean Field Limit of ρ_5

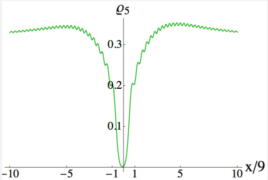
Scaling:

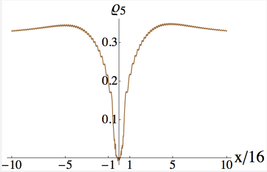
- $\widehat{m} \rightarrow s^2 \widehat{m}$
- $x \rightarrow s^2 x$
- $\widehat{a} \rightarrow s\widehat{a}$

$\widehat{m} = 3$, $\nu = 1$, $\widehat{a} = 0.5$
$s \in \{1, 2, 3, 4\}$







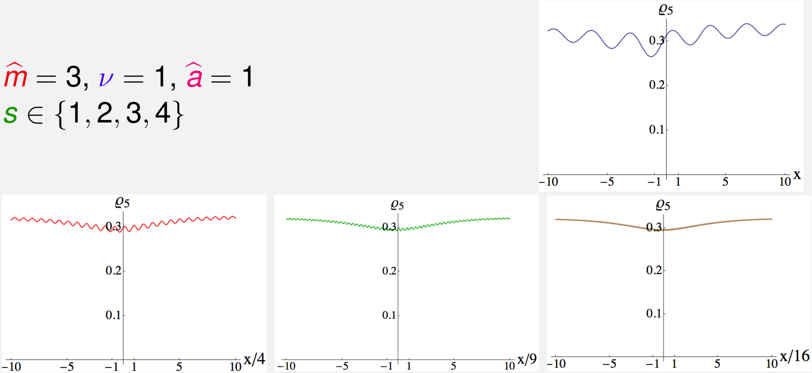


Mean Field Limit of ρ_5

Scaling:

- $\widehat{m} \to s^2 \widehat{m}$
- $x \to s^2 x$
- $\widehat{a} \to s\widehat{a}$

$\widehat{m} = 3,\ \nu = 1,\ \widehat{a} = 1$
$s \in \{1, 2, 3, 4\}$



ν Dependence of ρ_5 at large Lattice Spacing

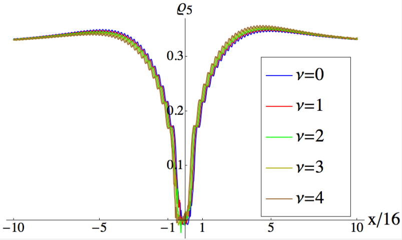
Scaling:

- $\widehat{m} \to s^2 \widehat{m}$
- $x \to s^2 x$
- $\widehat{a} \to s\widehat{a}$

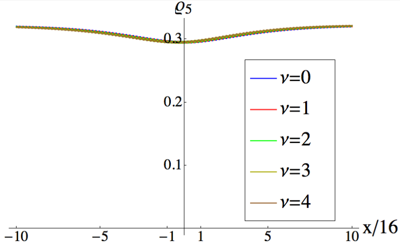
$\widehat{m} = 3$, $\widehat{a} \in \{0.5, 1\}$, $s = 4$

gap in $|x| \leq \left(\widehat{m}^{2/3} - 4\widehat{a}^{4/3}\right)^{3/2}$
$\Rightarrow$ gap only if $\widehat{m} > 8\widehat{a}^2$

not Aoki Phase



Aoki Phase



Damgaard, Splittorff, Verbaarschot (2010)