Random Matrix Theories for Dirac Operators at finite Lattice Spacing

M. Kieburg, J.J.M. Verbaarschot, and S. Zafeiropoulos

and by U.S. DOE Grant No. DE-FG-88ER40388

Introduction

In the low energy limit, QCD exhibits universal behavior that agrees with Random Matrix Theory (RMT) reflecting the symmetries of the Dirac operator. In the continuum limit as well as for finite lattice spacing $a \neq 0$ we have constructed random matrix ensembles with the eigenvalue correlations of QCD.

The models under consideration consist of matrices $D_{\nu}$, whose entries are weighted by independent Gaussian distributions. In the continuum limit, with lattice spacing $a = 0$, and quark mass $m = 0$, the chiral Gaussian Unitary Ensemble (chGUE) is defined by:

$$D_{\nu} = \begin{pmatrix} 0 & W \\ -W^T & 0 \end{pmatrix}, \text{ weighted by } P(D_{\nu}) \propto \exp(-\text{tr}(WW^T)), \quad (1)$$

where $W$ is a Gaussian complex matrix and $\nu$ is the topological charge. The matrix dimension plays the role of the lattice volume, i.e. $\nu \sim V$. The matrix $D_{\nu}$ reflects the chiral symmetry of the Dirac-operator in the continuum limit, i.e. $\gamma_5 D_{\nu} = -D_{\nu} \gamma_5$. For the eigenvalue density of this random matrix theory is the quenched case,

$$\rho(\nu) \propto \int d\nu P(D_{\nu}) \text{tr}(k^2 - D_{\nu}), \quad (2)$$

we find in the microscopic limit

$$\rho(\nu) \left( 1 - \frac{1}{\nu} \right)^{-3/2} \left( \frac{\nu}{\nu^c} \right)^{-1/2} + \frac{1}{\nu} \left( 1 - \frac{1}{\nu} \right)^{-3/2} \left( \frac{\nu}{\nu^c} \right)^{-1/2}$$

with $z \in \mathbb{R}$ and $J_\nu$ are Bessel functions of the first kind. This result also follows from the microscopic limit of the chiral Lagrangian for the Dirac spectrum. The Dirac-distribution corresponds to the $\nu$ generic zero eigenvalues.

The evol of the Wilson-Dirac Operator

For the Wilson-Dirac operator we consider the RMT

$$D_W = \begin{pmatrix} A & W \\ -W^T & B \end{pmatrix}, \text{ weighted by } P(D_W) \propto \exp(-\text{tr}(A A^T + \nu B B^T)), \quad (4)$$

where the $n \times n$ Hermitian matrix $A$ and the $(n + p) \times (n + p)$ Hermitian matrix $B$ break the chiral symmetry $[2-4]$. In the large $\nu$ limit we fix $2\nu m = \nu^2$ as a constant which is consistent with the $\nu$ counting scheme of Wilson chiral perturbation theory. We have derived analytical expressions for the eigenvalue density (evd) of the complex eigenvalues, $\rho_{C}$, and of the real eigenvalues, separately for the right- and left-handed eigenvectors, $\rho_{R}$ and $\rho_{L}$. The densities $\rho_{C}$ and $\rho_{L}$ are related by

$$\rho_C(z) = \rho_R(z) + \rho_L(z), \quad (5)$$

where $\rho_R$ is the chirality distribution. The distribution $\rho_{C, \nu^2}$ is the evo of the Gaussian unitary ensemble of dimension $\nu^2$. The interaction term $\rho_{\nu^2}$ accounts for the repulsion between the $\nu$ generic real eigenvalues with the remaining ones, and, thus, is not a distribution, i.e. not positive definite. The term $\rho_{\nu^2}$ vanishes for $\nu = 0$. For $\nu > 0$, we find

$$\rho_C(z) \propto \int \exp\left[ -\text{tr}\left( A A^T + z I + \nu B B^T \right) \right] \left( \frac{\nu}{\nu^c} \right)^{-3/2} \left( \frac{\nu}{\nu^c} \right)^{-1/2} \left( 1 - \frac{1}{\nu^c} \right)^{-3/2} \left( 1 - \frac{1}{\nu^c} \right)^{-1/2} + \frac{1}{\nu} \left( 1 - \frac{1}{\nu^c} \right)^{-3/2} \left( \nu^c / \nu^c \right)^{-1/2}.$$

The evol of the Dirac Operator for Staggered Fermions

We study a random matrix ensemble for the staggered Dirac operator that captures its main discretization effects. The model is defined by

$$D_{\alpha} = \begin{pmatrix} 0 & W + i \mathcal{A} \\ -W^T - i \mathcal{A}^T & aB \\ W - i \mathcal{A} \\ -W^T + i \mathcal{A}^T \end{pmatrix}, \text{ weighted by } P(D_{\alpha}) \propto \exp(-\text{tr}(W W^T + i \mathcal{A} \mathcal{A}^T + a B B^T)), \quad (10)$$

where $W, \mathcal{A}$ are $n \times (n + p)$ and $B, C$ are $(n + p) \times (n + p)$ and $n \times n$ complex matrices, respectively. It is a simplified version of the random matrix theory proposed in [5,6]. For $\alpha = 0$, this model has $2^{d/2}$ flavors for $d$ dimensions with zero modes for each flavor, while at $\alpha \neq 0$ we have flavor mixing and zero modes are absent. The spectral density shows chGUE behavior for $\alpha \ll 1/\sqrt{T}$ and becomes one chGUE when $\alpha \gg 1/\sqrt{T}$. The spectral flow with $\alpha$ shows avoided level crossings when lattice artifacts start dominating the Dirac spectrum. Monte Carlo simulations are shown in Fig. 3.

Conclusions

- **Wilson-Dirac Operator:** For $\nu > 1, \nu > 0$ gives most of the additional real eigenvalues. It behaves as $\nu^2$ whereas the number of real eigenvalues for $\nu \neq 0$ is suppressed by higher orders in $\nu$.
- **Staggered Fermions:** The scale $1/\sqrt{T}$ (i.e. $1/\sqrt{T}$ in lattice QCD) determines the appearance of lattice artifacts with increasing $\alpha$.

References