HOMEWORK 1, THERMAL PHYSICS (PHY306)

1. (a) Using data from chapter 2, calculate how much energy is needed to make a cup of tea (.2 kg of water, from $T = 20^oC$ to 100^oC).

(b) Assuming world energy production is $10^{13}W$, how much boiling water one can get in 1 year? Visualize it by assuming this water is making a cube of size L: how many meters such L will be? ($1\,m^3$ of water weights $10^3\,kg$.)

2.exercise (3.3)

3.exercise (3.5)

4.exercise (3.8) (a) and (b)

5. A model for rubber string is a one-dimensional chain of molecules, each with length a. Molecules are joined at their ends in a way, that the next one can go left or right with equal probability. If we denote n_+ the number going to the right and n_- the number going to the left, they satisfy the relations

$$n_+ + n_- = N, \quad X = a(n_+ - n_-)$$

where N is their total number and X is the length of the chain.
(a) what is the probability of having a given X, denoted by $W(X)$?
(b) Using Sterling's formula, calculate the entropy of the chain $S = k_B log(W)$
(c) Calculate the "entropic force"

$$F = -k_b T \frac{\partial S}{\partial X}$$

at small $x \ll Na$ and large $x \to Na$ length of the chain

1.) a.) $0.2\,kg\left(\frac{C_V}{m}\right)(100^\circ C - 20^\circ C)$

$\frac{C_V}{m}$ is the specific heat per unit mass

$\left(C_V = \left(\frac{\partial E}{\partial T}\right)_V = T\left(\frac{\partial S}{\partial T}\right)_V \right)$

b.) Power of Sun $= P = 10^{13} W$

$$\Delta E = P \underbrace{\Delta t}_{1 \text{ year}}$$

$$\frac{C_V}{V} = \left(\frac{C_V}{m}\right) \underbrace{\rho}_{\text{density}}$$

the energy need to boil 1 unit volume of water is

$$\frac{E_1}{V} = \left(\frac{C_V}{V}\right)(100°C - 20°C)$$

$$\frac{\Delta E}{\left(\frac{E_1}{V}\right)} = V\left(\frac{\Delta E}{E_1}\right)$$

is the volume of Boiling water

$$\frac{\Delta E}{\left(\frac{C_V}{m}\right)\rho(100°C - 20°C)} = \text{Volume of water}$$

2.) $\langle N \rangle = \frac{1}{6}\sum_{i=1}^{6} i$

$= \frac{1}{6} \cdot \frac{6(6+1)}{2}$

$= \frac{\cancel{3 \cdot 7}}{6} = 3.5 = \frac{7}{2}$

$\langle N^2 \rangle = \frac{1}{6}\sum_{i=1}^{6} i^2$

$= \frac{1}{6} \cdot \frac{n(n+1)(2n+1)}{6}$

$n = 6$

$= \frac{1}{6} \cdot 7 \cdot 13 = \frac{91}{6}$

$\delta N^2 = \frac{91}{6} - \frac{49}{4}$

7/12

$$= \frac{182}{12} - \frac{147}{12}$$

$$(\delta N)^2 = \frac{35}{12}$$

$$\Rightarrow \delta N = \sqrt{\frac{35}{12}}$$

Further reading

There are many good books on probability theory and statistics.
(2003), Wall and Jenkins (2003), and Sivia and Skilling (2006).

Exercises

(3.1) A throw of a regular die yields the numbers 1, 2, …, 6, each with probability 1/6. Find the mean, variance, and standard deviation of the numbers obtained.

(3.2) The mean birth-weight of babies in the UK is about 3.2 kg with a standard deviation of 0.5 kg. Convert these figures into pounds (lb), given that 1 kg = 2.2 lb.

(3.3) This question is about a *discrete* probability distribution known as the **Poisson distribution**. Let x be a discrete random variable that can take the values $0, 1, 2, \ldots$ A quantity x is said to be Poisson distributed if the probability $P(x)$ of obtaining x is

$$P(x) = \frac{e^{-m} m^x}{x!},$$

where m is a particular number (which we will show in part (b) of this exercise is the mean value of x).

(a) Show that $P(x)$ is a well-behaved probability distribution in the sense that

$$\sum_{x=0}^{\infty} P(x) = 1.$$

(Why is this condition important?)

(b) Show that the mean value of the probability distribution is $\langle x \rangle = \sum_{x=0}^{\infty} x P(x) = m$. (3.4)

(c) The Poisson distribution is useful for describing very rare events, which occur independently and whose average rate does not change over the period of interest. Examples include birth defects measured per year, traffic accidents at a particular junction per year, numbers of typographical errors on a page, and the number of activations of a Geiger counter per minute. The first recorded example of a

3.3) problem 2

$$\sum_{x=0}^{\infty} \frac{m^x e^{-m}}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{m^x}{x!}$$

$$= e^{-m} \cdot e^{m} = 1$$

$$\sum_{x=0}^{\infty} \frac{x m^{x} e^{-m}}{x!}$$

$$= m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$= m e^{-m} \sum_{x=0}^{\infty} \frac{m^{x}}{x!}$$

$$= m e^{-m} e^{m} = m$$

Further reading

There are many good books on probability theory and statistics. Recommended ones include Papoulis (1984), Saha (2003), Wall and Jenkins (2003), and Sivia and Skilling (2006).

Exercises

(3.1) A throw of a regular die yields the numbers 1, 2, ..., 6, each with probability 1/6. Find the mean, variance, and standard deviation of the numbers obtained.

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(b) Show that the mean value of the probability distribution is $\langle x \rangle = \sum_{x=0}^{\infty} x P(x) = m$.

(c) The Poisson distribution is useful for describing very rare events, which occur independently and whose average rate does not change over the period of interest. Examples include birth defects measured per year, traffic accidents at a particular junction per year, numbers of typographical errors on a page, and the number of activations of a Geiger counter per minute. The first recorded example of a Poisson distribution, the one which in fact motivated Poisson, was connected with the rare event of someone being kicked to death by a horse in the Prussian army. The number of horse-kick deaths of Prussian military personnel was recorded for each of 10 corps in each of 20 years from 1875–1894 and the following data recorded:

Number of deaths per year, per corps	Observed frequency
0	109
1	65
2	22
3	3
4	1
≥ 5	0
Total	200

Calculate the mean number of deaths per year per corps. Compare the observed frequency with a calculated frequency assuming the number of deaths per year per corps are Poisson distributed with this mean.

(3.4) This question is about a *continuous* probability distribution known as the **exponential distribution**. Let x be a continuous random variable that can take any value $x \geq 0$. A quantity is said to be exponentially distributed if it takes values between x and $x + dx$ with probability

$$P(x)\,dx = Ae^{-x/\lambda}\,dx,$$

where λ and A are constants.

(a) Find the value of A that makes $P(x)$ a well-defined continuous probability distribution so

2) c.)

$$M = 0 \cdot \frac{104}{200} + 1 \cdot \frac{65}{200} + 2 \cdot \frac{22}{200}$$

+3

$$+ 3 \cdot \frac{3}{200} + 4 \cdot \frac{1}{200}$$

$$= 0.61$$

$$P(x) = \frac{m^x e^{-m}}{x!}$$

X	P(x)	actual freq.
0	0.543	104/200 = 0.545
1	0.331	65/200 = 0.325
2	0.101	22/200 = 0.11
3	0.0206	3/200 = 0.015

4	0.00313	1/200 = 0.005
≥5	0.00127	0/200 = 0

+3

3.)

Prob 3.5

that $\int_0^\infty P(x)\,\mathrm{d}x = 1$.

(b) Show that the mean value of the probability distribution is $\langle x\rangle = \int_0^\infty xP(x)\,\mathrm{d}x = \lambda$.

(c) Find the variance and standard deviation of this probability distribution. Both the exponential distribution and the Poisson distribution are used to describe similar processes, but for the exponential distribution x is the *actual time* between, for example, successive radioactive decays, successive molecular collisions, or successive horse-kicking incidents (rather than, as with the Poisson distribution, x being simply *the number* of such events in a specified interval).

(3.5) If θ is a continuous random variable which is uniformly distributed between 0 and π, write down an expression for $P(\theta)$. Hence find the value of the following averages:

(a) $\langle\theta\rangle$;
(b) $\langle\theta - \frac{\pi}{2}\rangle$;
(c) $\langle\theta^2\rangle$;
(d) $\langle\theta^n\rangle$ (for the case $n \geq 0$);
(e) $\langle\cos\theta\rangle$;
(f) $\langle\sin\theta\rangle$;
(g) $\langle|\cos\theta|\rangle$;
(h) $\langle\cos^2\theta\rangle$;
(i) $\langle\sin^2\theta\rangle$;
(j) $\langle\cos^2\theta + \sin^2\theta\rangle$.

Check that your answers are what you expect.

(3.6) In experimental physics, it is important to repeat measurements. Assuming that errors are random, show that if the error in making a single measurement of a quantity X is Δ, the error obtained after using n measurements is $\Delta/\sqrt{n}$. (Hint: after n measurements, the procedure would be to take the n results and average them. So you require the standard deviation of the quantity $Y = (X_1+X_2+\cdots+X_n)/n$ where $X_1, X_2, \ldots, X_n$ can be assumed to be independent, and each has standard deviation Δ.)

(3.7) **(a)** Show that the binomial distribution can be approximated by a Poisson distribution with mean np when $n \gg 1$ but np remains small. (This therefore represents the case when $p \ll 1$ so that "success" is a rare event.)

(b) A harder problem is to show that when $n \gg 1$ and also $np(1-p) \gg 1$ the binomial distribution can be approximated by a Gaussian distribution with mean np and variance $np(1-p)$. Assuming this to be the case, revisit the one-dimensional random walk in Example 3.10 and assume that the walker takes a step when time $t = n\tau$, where n is an integer. Writing $D = L^2/2\tau$ and using eqns 3.48 and 3.49 show that when $t \gg \tau$ the probability of finding the particle between x and $x + \mathrm{d}x$ is

$$P(x)\,\mathrm{d}x = \frac{1}{\sqrt{4\pi Dt}}\mathrm{e}^{-x^2/4Dt}\,\mathrm{d}x. \qquad (3.50)$$

[See also Appendix C.12 for an alternative derivation of eqn 3.50.]

(c) Show that the standard deviation of the distribution in eqn 3.50 is given by $\sigma_x = \sqrt{2Dt}$. As the random walker "diffuses" backwards and forwards, you could try and define its diffusion speed by σ_x/t. This gives a speed that is proportional to $t^{-1/2}$ and is clearly nonsense. The point about diffusion (the behaviour of random walkers) is that since $\sigma_x \propto t^{1/2}$ you need 100 times as much time to diffuse a distance 10 times as big. A small molecule in water diffuses at a rate governed by $D = 10^{-9}\,\mathrm{m^2\,s^{-1}}$. Estimate the time needed for this molecule to diffuse about (i) $1\,\mu$m (the width of a bacterium) and (ii) 1 cm (the width of a test tube).

(3.8) This question introduces a rather efficient method for calculating the mean and variance of probability distributions. We define the **moment generating function** $M(t)$ for a random variable x by

$$M(t) = \langle\mathrm{e}^{tx}\rangle. \qquad (3.51)$$

Show that this definition implies that

$$\langle x^n\rangle = M^{(n)}(0), \qquad (3.52)$$

where $M^{(n)}(t) = \mathrm{d}^nM/\mathrm{d}t^n$ and further that the mean $\langle x\rangle = M^{(1)}(0)$ and the variance $\sigma_x = M^{(2)}(0) - [M^{(1)}(0)]^2$. Hence show that:

(a) for a single Bernoulli trial,

$$M(t) = p\mathrm{e}^t + 1 - p; \qquad (3.53)$$

(b) for the binomial distribution,

$$M(t) = (p\mathrm{e}^t + 1 - p)^n; \qquad (3.54)$$

(c) for the Poisson distribution,

$$M(t) = \mathrm{e}^{m(\mathrm{e}^t-1)}; \qquad (3.55)$$

(d) for the exponential distribution,

$$M(t) = \frac{\lambda}{\lambda - t}. \qquad (3.56)$$

Hence derive the mean and variance in each case and show that they agree with the results derived earlier.

$$P(\theta) = \frac{1}{\pi}$$

$$a.)\ \langle\theta\rangle = \int_0^\pi d\theta\ \theta P(\theta) = \frac{1}{\pi}\cdot\frac{\pi^2}{2}$$

$$= \frac{\pi}{2}$$

b.) $\langle \theta - \frac{\pi}{2} \rangle = \langle \theta \rangle - \frac{\pi}{2} = 0$

c.) $\langle \theta^2 \rangle = \int_0^{\pi} d\theta \, \frac{\theta^2}{\pi} = \frac{1}{\pi} \cdot \frac{\pi^3}{3}$

$= \frac{\pi^2}{3}$

d.) $\langle \theta^n \rangle = \int_0^{\pi} d\theta \, \frac{\theta^n}{\pi} = \frac{\pi^{n+1}}{(n+1)\pi}$

$= \frac{\pi^n}{n+1}$

e.) $\langle \cos\theta \rangle = \frac{1}{\pi}\int_0^{\pi} d\theta \cos\theta = 0$

f.) $\langle \sin\theta \rangle = \frac{1}{\pi}\int_0^{\pi} d\theta \sin\theta = \frac{2}{\pi}$

g.) $\langle |\cos\theta| \rangle = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} d\theta \cos\theta = \frac{2}{\pi}$

h.) $\langle \cos^2\theta \rangle = \frac{1}{\pi}\int_0^{\pi} d\theta \, \frac{1}{2} + \cancelto{0}{\frac{1}{2}\cos(2\theta)} = \frac{1}{2}$

i.) $\langle \sin^2\theta \rangle = \frac{1}{\pi}\int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2\theta) d\theta = \frac{1}{2}$

(j) $\langle \cos^2\theta + \sin^2\theta \rangle = \langle \cos^2\theta \rangle + \langle \sin^2\theta \rangle$

$$= \frac{1}{2} + \frac{1}{2} = 1$$

4.7

Prob 3.8

cube).

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Hence derive the mean and variance in each case and show that they agree with the results derived earlier.

First part of the problem

$$M(t) = \langle e^{tx} \rangle$$

$$\Rightarrow M(t) = \int dx\, P(x)\, e^{tx}$$

$$= \sum_{k=0}^{\infty} \frac{t^k}{k!} \left[\int dx\, P(x)\, x^k \right]$$

$$\frac{\partial^n M}{\partial t^n} = \sum^{\infty} \frac{k!\, t^{k-n}}{(k-n)!\, k!} \int dx\, P(x)\, x^k$$

∂t^{i} $\quad k=n$

$$= \sum_{j=0}^{\infty} \frac{t^{j}}{j!} \int dx\, P(x)\, X^{j+n}$$

$[\, j = k-n, \quad k = j+n \,]$

if $t=0$ then

$$t^{j} = \begin{cases} 1 & \text{if } j=0 \\ 0 & \text{if } j>0 \end{cases}$$

$$\Rightarrow \left.\frac{\partial^{n} M}{\partial t^{n}}\right|_{t=0} = \int dx\, P(x)\, X^{0+n} = \int dx\, P(x)\, X^{n}$$

$$\equiv \langle X^{n} \rangle$$

Based on this

$$\left.\frac{\partial^{1} M}{\partial t^{1}}\right|_{t=0} = M^{(1)}(0) = \langle X^{1} \rangle = \langle x \rangle$$

and

$$M^{(2)}(0) = \langle X^{2} \rangle$$

Since $\sigma_{x}^{2} = \langle X^{2} \rangle - \langle X \rangle^{2}$

substitution w/ M functions give

$$\sigma_X^2 = M^{(2)}(0) - (M^{(1)}(0))^2$$

a.) **(a)** for a single Bernoulli trial,

$$M(t) = pe^t + 1 - p; \qquad (3.53)$$

$\int dx\, P(x) \longleftrightarrow \sum_{x_i} P(x_i)$ for discreet distributions

$$M(t) \equiv \langle e^{tx} \rangle$$

$$= \sum_{x=0}^{1} P(x)\, e^{tx}$$

$$= P(0)\, e^{0} + P(1)\, e^{1 \cdot x}$$

$P(0) = 1 - p \qquad e^{0} = 1$

$P(1) = p \qquad e^{1 \cdot x} = e^{x}$

$$\Rightarrow M(t) = (1-p) + p e^{x}$$

b.) **(b)** for the binomial distribution,

$$M(t) = (pe^t + 1 - p)^n; \qquad (3.54)$$

$$M(t) = \sum_{x=0}^{n} \binom{n}{x} p^x (1-p)^{n-x} e^{tx}$$

$$= \sum_{x=0}^{n} \binom{n}{x} (pe^t)^x (1-p)^{n-x}$$

using the fact that

$$(a+b)^n = \sum_{x=0}^{n} \binom{n}{x} a^x b^{n-x}$$

then $a = pe^t$, $b = 1-p$

so

$$M(t) = (pe^t + 1 - p)^n$$

c.)

(c) for the Poisson distribution,

$$M(t) = \mathrm{e}^{m(\mathrm{e}^t-1)}; \qquad (3.55)$$

$$M(t) = \sum_{x=0}^{\infty} \frac{m^x e^{xt} e^{-m}}{x!}$$

$$= e^{-m} \sum_{x=0}^{\infty} \frac{(me^t)^x}{x!}$$

$$= e^{-m} \cdot e^{me^t} = e^{m(e^t - 1)}$$

d.) **(d)** for the exponential distribution,

$$M(t) = \frac{\lambda}{\lambda - t}. \qquad (3.56)$$

the Probability density function of an exponential dist. is

$$P(x) = \begin{cases} \lambda e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

$$= \lambda e^{-\lambda x} \Theta(x)$$

where Θ is the heaviside step function.

$$M(t) = \int dx\, P(x)\, e^{tx}$$

$$= \lambda \int_0^\infty dx\, e^{-\lambda x} e^{tx} = \lambda \int_0^\infty dx\, e^{(t-\lambda)x}$$

$$= \lambda \left[\frac{e^{(t-\lambda)x}}{t-\lambda} \right]_0^\infty$$

assuming $t < \lambda$

we get

$$M(t) = \lambda\left(\frac{0}{t-\lambda} - \frac{1}{t-\lambda}\right)$$

$$= \frac{\lambda}{\lambda - t} \qquad \blacksquare$$

5. A model for rubber string is a one-dimensional chain of molecules, each with length a. Molecules are joined at their ends in a way, that the next one can go left or right with equal probability. If we denote n_+ the number going to the right and n_- the number going to the left, they satisfy the relations

$$n_+ + n_- = N, \quad X = a(n_+ - n_-)$$

where N is their total number and X is the length of the chain.
(a) what is the probability of having a given X, denoted by $W(X)$?
(b) Using Sterling's formula, calculate the entropy of the chain $S = k_B log(W)$
(c) Calculate the "entropic force"

$$F = -k_b T \frac{\partial S}{\partial X}$$

at small $x \ll Na$ and large $x \to Na$ length of the chain

Since the chain has an equal probability of going either direction then

$$P(+) = \frac{1}{2} = P(-)$$

Since in this case $P(+)$ can represent a "success" while $P(-)$ is a "failure" in Bernoulli terms. Since there are N atoms (trials) then the distribution of length is a Binomial distribution.

$$P(n_+) = \binom{N}{n_+}\left(\frac{1}{2}\right)^{n_+}\left(\frac{1}{2}\right)^{N-n_+}$$

or

$$P(n_+, n_-) = \frac{N!}{n_+!\, n_-!}\left(\frac{1}{2}\right)^{n_+}\left(\frac{1}{2}\right)^{n_-}$$

and $N = n_+ + n_-$

a.) $X \equiv a(n_+ - n_-)$

$$\langle X \rangle = \langle a(n_+ - n_-) \rangle$$
$$= a\langle n_+ - n_- \rangle = a\langle n_+ \rangle - a\langle n_- \rangle$$

Since n_+, n_- have a sym. of Probability we will expect

$\langle n_+ \rangle = \langle n_- \rangle \Rightarrow \langle X \rangle = 0$

Now lets Prove that

Our generating function in 3b is

$$M(t) = (pe^t + 1 - p)^n$$

and $\frac{\partial^k M}{\partial t^k}\Big|_{t=0} = \langle x^k \rangle$

Since $p = 1/2$, $X = n_+$

then

$$\langle n_+ \rangle = \frac{d}{dt}\left(\tfrac{1}{2}\right)^N (e^t + 1)^N \Big|_{t=0}$$

$$= \left(\tfrac{1}{2}\right)^N N (e^t + 1)^{N-1} \cdot e^t \Big|_{t=0}$$

$$= \left(\tfrac{1}{2}\right)^N \cdot N(2)^{N-1}$$

$$= \frac{N}{2}$$

$$\langle n_- \rangle = \langle N \rangle - \langle n_+ \rangle$$

$$= N\langle 1 \rangle - \frac{N}{2}$$

$$= N \underbrace{\left(\tfrac{1}{2} + \tfrac{1}{2}\right)^N}_{M(0) = \left.\frac{\partial^0 M}{\partial t^0}\right|_{t=0}} - \frac{N}{2}$$

$$\Rightarrow \langle n_- \rangle = N - \frac{N}{2} = \frac{1}{2} N$$

thus

$$\langle X \rangle = a \langle n_+ \rangle - a \langle n_- \rangle$$

$$= a \cdot \frac{N}{2} - a \frac{N}{2} = 0$$

b.) $X = a(n_+ - n_-) = a(n_+ - (N - n_+))$

$$= a(2n_+ - N)$$

thus the # of ways to get a particular X is dependent on the # of ways to get n_+ which is $\binom{N}{n_+}$.

Putting n_+ in terms of x we get

$$\frac{1}{2}\left(\frac{x}{a}+N\right) = n_+$$

$$\Rightarrow \binom{N}{\left(\frac{1}{2}\left(\frac{x}{a}+N\right)\right)} = \frac{N!}{\left(\frac{1}{2}\left(\frac{x}{a}+N\right)\right)!\,\left(\frac{1}{2}\left(N-\frac{x}{a}\right)\right)!}$$

$\Rightarrow$

$$S = k_B\left[\ln N! - \ln\left(\left(\frac{1}{2}\left(N+\frac{x}{a}\right)\right)!\right) - \ln\left(\left(\frac{1}{2}\left(N-\frac{x}{a}\right)\right)!\right)\right]$$

Sterling's formula is

$$\ln(j!) = j\ln j - j$$

$$\Rightarrow S = k_B\Big[N\ln N - \frac{1}{2}\left(N+\frac{x}{a}\right)\ln\left(\frac{1}{2}\left(N+\frac{x}{a}\right)\right) - \frac{1}{2}\left(N-\frac{x}{a}\right)\ln\left(\frac{1}{2}\left(N-\frac{x}{a}\right)\right) - N + \frac{1}{2}\left(N+\frac{x}{a}\right) + \frac{1}{2}\left(N-\frac{x}{a}\right)\Big]$$

then simplified too

$$S = k_B\left[N\ln N - \frac{1}{2}\left(N+\frac{x}{a}\right)\ln\left(\frac{1}{2}\left(N+\frac{x}{a}\right)\right) - \frac{1}{2}\left(N-\frac{x}{a}\right)\ln\left(\frac{1}{2}\left(N-\frac{x}{a}\right)\right)\right]$$

(.) $$F = -T\frac{\partial S}{\partial x}$$

$$\frac{\partial S}{\partial x} = \Big[-\frac{1}{a}\ln\left(\tfrac{1}{2}\left(N + \tfrac{x}{a}\right)\right) + \frac{1}{a}\ln\left(\tfrac{1}{2}\left(N - \tfrac{x}{a}\right)\right) - \frac{\frac{1}{2}\left(N + \frac{x}{a}\right)}{\frac{1}{2}\left(N + \frac{x}{a}\right)}\cdot\frac{1}{2a} - \frac{\frac{1}{2}\left(N - \frac{x}{a}\right)}{\frac{1}{2}\left(N - \frac{x}{a}\right)}\cdot\left(-\frac{1}{2a}\right)\Big] k_B$$

$$= \frac{k_B}{a}\ln\left(\frac{N - \frac{x}{a}}{N + \frac{x}{a}}\right)$$

thus

$$F = \frac{k_B T}{a}\ln\left(\frac{N + \frac{x}{a}}{N - \frac{x}{a}}\right)$$

$$= \frac{k_B T}{a}\ln\left(\frac{Na + x}{Na - x}\right)$$

$$= \frac{k_B T}{a}\ln\left(\frac{1 + \frac{x}{Na}}{1 - \frac{x}{Na}}\right)$$

$\left(1 - \frac{x}{Na}\right)$

if $x \ll Na \Rightarrow \frac{x}{Na} \ll 1$

so $$\frac{1+\frac{x}{Na}}{1-\frac{x}{Na}} \approx \left(1+\frac{x}{Na}\right)\underbrace{\left(1+\frac{x}{Na}+\left(\frac{x}{Na}\right)^2+\cdots\right)}_{\text{Geometric Series}}$$

$$= 1 + \frac{2x}{Na} + O\left(\left(\frac{x}{Na}\right)^2\right)$$

thus

$$F \approx \frac{k_BT}{a}\ln\left(1+\frac{2x}{Na}\right)$$

$$\ln(1+\varepsilon) = 0 + \varepsilon - \frac{\varepsilon^2}{2} \cdots$$

so

$$F \approx \frac{k_BT}{a}\left(\frac{2x}{Na}\right) = \frac{2k_BTx}{Na^2}$$

if $x \to Na$ then $\frac{x}{Na} \to 1$

$$\lim_{\varepsilon \to 1^-} \ln\left(\frac{1+\varepsilon}{1-\varepsilon}\right) = \infty$$

So $F \to \infty$ as $X \to Na$

A possible reason is that our constraints assume that the longest length is Na. IN order to go past the longest length one needs to create a nonexistent state.