HOMEWORK 1, THERMAL PHYSICS (PHY306)

1. (a) Using data from chapter 2, calculate how much energy is needed to make a cup of tea (.2 kg of water, from $T = 20^{\circ}C$ to $100^{\circ}C$).

(b) Assuming world energy production is $10^{13}W$, how much boiling water one can get in 1 year? Visualize it by assuming this water is making a cube of size L: how many meters such L will be? $(1 m^3 \text{ of water weights } 10^3 kg.)$

2.exercise (3.3)

3.exercise (3.5)

4.exercise (3.8) (a) and (b)

5. A model for rubber string is a one-dimensional chain of molecules, each with length a. Molecules are joined at their ends in a way, that the next one can go left or right with equal probability. If we denote n_+ the number going to the right and n_- the number going to the left, they satisfy the relations

$$n_+ + n_- = N$$
, $X = a(n_+ - n_-)$

where N is their total number and X is the length of the chain.

(a) what is the probability of having a given X, denoted by W(X)?

(b) Using Sterling's formula, calculate the entropy of the chain $S = k_B log(W)$

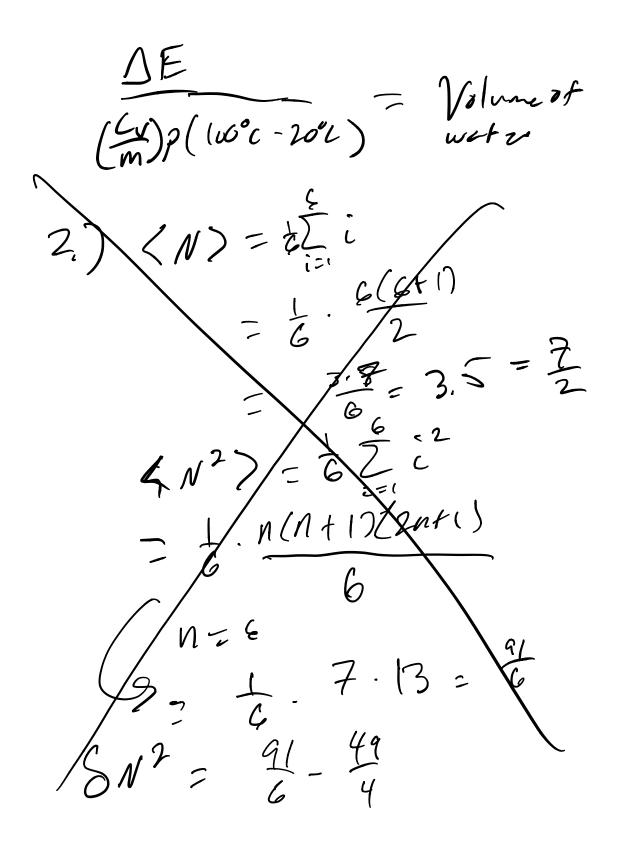
(c) Calculate the "entropic force"

$$F = -k_b T \frac{\partial S}{\partial X}$$

at small $x \ll Na$ and large $x \to Na$ length of the chain

$$\begin{array}{l} (1) \quad r. \end{array}) \quad 0.2 \, k_{3} \left(\frac{(v)}{m} \right) \left(100 \, e^{-20 \, v} \right) \\ \frac{C v}{m} \quad is \quad the \quad Specific het \\ fer \quad vnit \quad m-ss \\ \left(C v = \left(\frac{\partial E}{\partial T} \right)_{v} = \pm \left(\frac{\partial S}{\partial r} \right)_{v} \end{array} \right) \end{array}$$

- P=100 5.) Power of Sun AE = PAt Year $\frac{Cv}{V} = (\frac{Cv}{m})P$ the every Dersity need to boil 1 orit volum Of water $\frac{E_{i}}{V} = \frac{V}{V} \left(\frac{100}{100} \left(-20^{\circ} L \right) \right)$ $\begin{array}{c}
AE = V\left(AE \\
E\end{array}\right) \\
\left(EV\right)
\end{array}$ is the Volum of Boiling water



= 182 [2 147 (22 35

Further reading

There are many good books on probability theory and statistics. (2003), Wall and Jenkins (2003), and Sivia and Skilling (2006).

Exercises

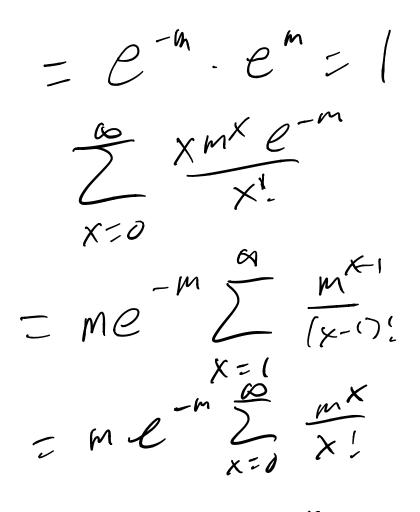
- EXELEMENT
 (3). A three of a regular dis yields the numbers 1, 2, ..., 6, and with probability 1/6. Find the mean, variance, and standard deviation of the numbers obtained.
 (3). The mean little weight of babies in the UK is about 3.2 kg with a standard deviation of 0.5 kg. Covert the 2 mean figures into portion [10], your hard 1 kg = figures into portion [10], your hard 1 kg = the a discrete random wavelet that the the value of 0.1 μg. ..., A quantity is a solid to be basin distribution. Let be a discrete random wavelet that the inter when 0.12..., A quantity is valid to be basin distribution. Here, ..., A quantity is valid to be basin distribution. Here, ..., a quantity is valid to be basin distribution. Here, ..., A quantity is valid to be basin.

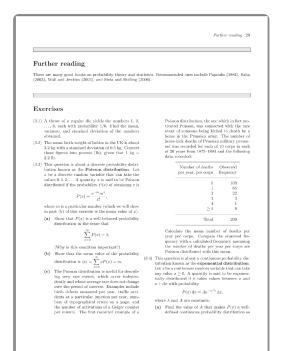
$P(x) = \frac{\mathrm{e}^{-m}m^x}{x!},$

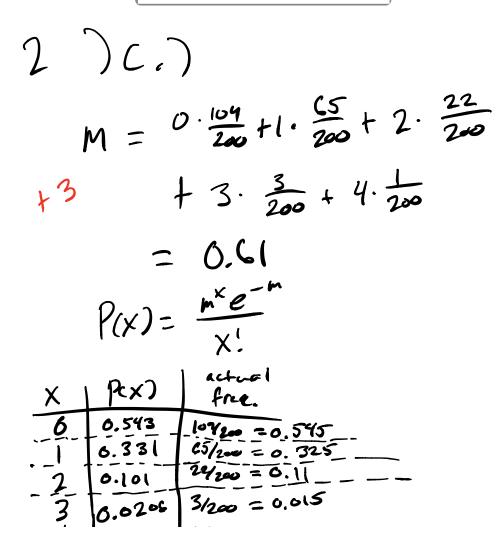
x!
x!
where m is a particular number (which we will show in part (b) of this exercise is the mean value of x).
(a) Show that P(x) is a well-behaved probability distribution in the sense that

- distribution in the sense that $\sum_{n=0}^{\infty} P(x) = 1.$ (Why is this contain important?) (b) Show that the mean value of the probability distribution is $\langle x \rangle = \sum_{n=0}^{\infty} x P(x) = m.$ (3.4) (c) The Possen distribution, useful for discrib-denty and whose average rate to do not dinage over the period of interest. Examples include litth disfect measured are year, traffic acci-dently and whose average rate does not dinage over the period of interest. Examples include litth disfect measured are year, traffic acci-bilith disfect measured are year, traffic acci-bilith disfect of activitions of a Geoger counter

m m×4 Proble x=0 _ p 200 M







4 0.00313 1/200 = 0.005 25 0.00127 0/200 = 0 + 3

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30 Exercises

- that $\int_0^\infty P(x) \, \mathrm{d}x = 1.$
- that ∫₀[∞] P(x) dx = 1.
 (b) Show that the mean value of the probability distribution is ⟨x⟩ = ∫₀[∞] xP(x) dx = λ.
 (c) Find the variance and standard deviation of this probability distribution. Both the exponential distribution and the Poisson distribution are used to describe similar processes, but for the exponential distribution x is the actual time between, for example, successive radioactive decays, successive molecular collisions, or successive horse-kicking incidents (rather than, as with the Poisson distribution, x being simply the number of such events in a specified interval).
) If θ is a continuous random variable which is uni-
- (3.5) If θ is a continuous random variable which is uniformly distributed between 0 and π , write down an expression for $P(\theta)$. Hence find the value of the following averag (a) $\langle \theta \rangle$; (b) $\langle \theta - \frac{\pi}{2} \rangle$; (c) $\langle \theta^2 \rangle$;

 - (d) $\langle \theta^n \rangle$ (for the case $n \ge 0$);
 - (e) $\langle \cos \theta \rangle$;
 - (c) $\langle \cos \theta \rangle$; (f) $\langle \sin \theta \rangle$; (g) $\langle |\cos \theta| \rangle$; (h) $\langle \cos^2 \theta \rangle$;

 - (i) $\langle \sin^2 \theta \rangle$; (j) $\langle \cos^2 \theta + \sin^2 \theta \rangle$.
- (i) (cos² θ + sin² θ). Check that your answers are what you expect.
 (3.6) In experimental physics, it is important to repeat measurements. Assuming that errors are random, show that if the error in making a single measure-ment of a quantity X is Δ, the error obtained af-ter using m measurements is Δ/√n. (Mint. af-ter n measurements, the procedure would be to take the n results and average them. So you re-quire the standard deviation of the quantity Y = (X₁+X₂+...+X_n)/n where X₁, X₂,..., X_n can be assumed to be independent, and each has standard deviation Δ.)
- deviation Δ.)
 (3.7) (a) Show that the binomial distribution can be approximated by a Poisson distribution with mean np when n ≥ 1 but np remains small. (This therefore represents the case when p ≪ 1 so that "success" is a rare event.)
 (b) A harder problem its obsove that when n ≫ 1 and also np(1 p) ≫ 1 the binomial distribution with mean np and variance np(1 p). Assuming this to be the case, neverish one dimensional random walk in Example 3.10 and assume that the walker takes a

	step when time $t = n\tau$, where n is an integer. Writing $D = L^2/2\tau$ and using eqns 3.48 and 3.49 show that when $t \gg \tau$ the probability of finding the particle between x and $x + dx$ is				
	$P(x) \mathrm{d}x = \frac{1}{\sqrt{4\pi Dt}} \mathrm{e}^{-x^2/4Dt} \mathrm{d}x. \tag{3.50}$				
	[See also Appendix C.12 for an alternative derivation of eqn 3.50.]				
(0.0)	(c) Show that the standard deviation of the distribution in eqn 3.50 is given by σ _α = √2Dτ. As the random walker "diffuses" backwards and forwards, you could try and define its diffusion speed by σ _α /t. This gives a speed that is proportional to t ^{-1/2} and is clearly monsense. The point about diffusion (the behaviour of random walkers) is that since σ _α ∝ t ^{-1/2} you need 100 times as much time to diffuse a distance 100 times as big. A small molecule in water diffuses at a rate governed by D = 10 ⁻¹⁰ m ² s ⁻¹ . Estimate the time needed for this molecule to diffuse about (i) 1 µm (the width of a bacterium) and (ii) 1 cm (the width of a test tube).				
(3.8)	This question introduces a rather efficient method for calculating the mean and variance of probability distributions. We define the moment generating function $M(t)$ for a random variable x by				
	$M(t) = \langle e^{tx} \rangle.$ (3.51)				
	Show that this definition implies that				
	$\langle x^n \rangle = M^{(n)}(0),$ (3.52)				
	where $M^{(n)}(t) = d^n M/dt^n$ and further that the mean $\langle x \rangle = M^{(1)}(0)$ and the variance $\sigma_x =$ $M^{(2)}(0) - [M^{(1)}(0)]^2$. Hence show that:				
	(a) for a single Bernoulli trial,				
	$M(t) = pe^{t} + 1 - p;$ (3.53)				
	(b) for the binomial distribution,				
	$M(t) = (pe^{t} + 1 - p)^{n};$ (3.54)				
	(c) for the Poisson distribution,				
	$M(t) = e^{m(e^t - 1)};$ (3.55)				
	(d) for the exponential distribution,				
	$M(t) = \frac{\lambda}{\lambda - t}.$ (3.56)				
	Hence derive the mean and variance in each case and show that they agree with the results derived earlier.				

 $P(G) = \overline{\pi}_{\pi}$ $a.) \langle e \rangle = \int_{a}^{\pi} \int_{a} G G P(G) = \frac{1}{\pi}$ Ö

$$b) \langle 6 - \frac{\pi}{2} \rangle = \langle 6 \rangle - \frac{\pi}{2} = 0$$

$$c_{0} \langle 6^{2} \rangle = \int_{0}^{11} \frac{e}{4\pi} = \frac{1}{\pi} \cdot \frac{\pi^{3}}{3}$$

$$= \frac{\pi^{2}}{3}$$

$$d_{0} \rangle \langle 6^{n} \rangle = \int_{0}^{11} \frac{e}{4\pi} = \frac{\pi^{n+1}}{(n+1)\pi}$$

$$= \frac{\pi^{n}}{n+1}$$

$$e_{0} \langle (056) = \frac{1}{2} \int_{0}^{11} \frac{e}{6056} = 0$$

$$f_{0} \rangle \langle (056) = \frac{1}{2} \int_{0}^{11} \frac{e}{6056} = \frac{2}{\pi}$$

$$f_{0} \rangle \langle (056) = \frac{1}{2} \int_{0}^{11} \frac{e}{6056} = \frac{2}{\pi}$$

$$h_{0} \rangle \langle (056) = \frac{1}{2} \int_{0}^{11} \frac{e}{6056} = \frac{1}{2}$$

$$i_{0} \rangle \langle (056) = \frac{1}{2} \int_{0}^{11} \frac{1}{2} \frac{e}{6066} = \frac{1}{2}$$

$$i_{0} \rangle \langle (05^{2}6) = \frac{1}{2} \int_{0}^{11} \frac{1}{2} \frac{e}{6056} = \frac{1}{2}$$

$$(j) \langle (05^{2}6) = \frac{1}{2} \int_{0}^{11} \frac{1}{2} \frac{e}{6056} + \frac{1}{2} \frac{1}{2} \frac{e}{6056} + \frac{1}{2} \frac{1}{2} \frac{e}{6056} + \frac{1}{2} \frac{1}{2} \frac{1}{2} \frac{e}{6056} + \frac{1}{2} \frac{1$$

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Show that this definition implies that

$$\langle x^n \rangle = M^{(n)}(0),$$

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(3.56)

where $M^{(n)}(t) = d^n M/dt^n$ and further that the mean $\langle x \rangle = M^{(1)}(0)$ and the variance $\sigma_x = M^{(2)}(0) - [M^{(1)}(0)]^2$. Hence show that:

(a) for a single Bernoulli trial,

	$M(t) = p e^{t}$	+1-p;	(3.53)
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(b) for the binomial distribution,	
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$$M(t) = (pe^{t} + 1 - p)^{n}; \qquad (3.54)$$
(c) for the Poisson distribution,

$$M(t) = e^{m(e^t - 1)}; (3.55)$$

(d) for the exponential distribution,

$$M(t) = \frac{\lambda}{\lambda - t}.$$

Hence derive the mean and variance in each case and show that they agree with the results derived earlier.

First Part of the problem M(E) = < et Z

τX => $M(t) = \int dx P(x) e$ = ZEKTSAKPCX)X" $\frac{\partial M}{\partial M} = \sum_{(k-N): k} \frac{k! t^{k-n}}{k! t^{k-n}} \int dx \mathcal{R}_{x} \mathcal{X}^{k}$

$$\begin{array}{l} \partial t & k=n \\ &= \displaystyle\sum_{j=0}^{\infty} \frac{t^{j}}{j!} \int dx f x j x^{j+n} \\ & E \, j = k-n, \ k = j+n] \\ if \ t = 0 \ then \\ t^{0} &= \displaystyle\sum_{j=0}^{j-1} if \ j = 0 \\ t^{0} &= \displaystyle\sum_{j=0}^{j-1} if \ j = 0 \\ &= \displaystyle\sum_{j=0}^{j-1} \partial x^{j+1} \Big|_{t=0} \\ &= \displaystyle\sum_{j=0}^{j-1} \partial x f x^{j+1} = \displaystyle\int dx f x x x^{j+1} \\ &= \displaystyle\int dx f x x^{j+1} \\ &= \displaystyle\int dx f x^$$

۵) (a) for a single Bernoulli trial, $M(t) = pe^t + 1 - p;$ (3.53)(JAXPCX) ~ Z P(K) for X: discreet Nistributions $M(\epsilon) = \langle e^{\epsilon x} \rangle$ $= \int_{-}^{+} P(x) e^{tx}$ $= P(o)e^{i} + P(i)e^{i}$ X = 0 $P(0) = 1 - P e^{2} = ($ $P(1) = \rho \quad e^{i \cdot x} = e^{x}$ =) $M(t) = (1-P) + Pe^{t}$

(b) for the binomial distribution, $M(t) = (pe^{t} + 1 - p)^{n}; \quad (3.54)$

b.)

$$M(t) = \sum_{x=0}^{n} {\binom{n}{x}} p^{x} (1-p)^{n \cdot x} e^{t \cdot x}$$

$$= \sum_{x=0}^{n} {\binom{n}{x}} (pe^{t})^{x} (1-p)^{n \cdot x}$$

$$x=0$$

$$(asing the fact that (a+b)^{n} = \sum_{x=0}^{n} {\binom{n}{x}} a^{x} b^{n-x}$$

$$fhen a = fe^{t}, b = f-p$$

$$So$$

$$M(t) = (pe^{t} + (-p)^{n})$$

(c) for the Poisson distribution,

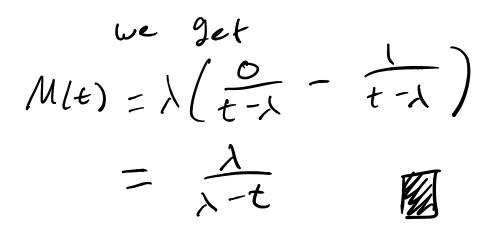
$$M(t) = e^{m(e^{t}-1)}; \quad (3.55)$$

$$M(t) = \sum_{X=0}^{\infty} \frac{m^{X} e^{xt} e^{-m}}{X^{\frac{1}{2}}}$$

$$= e^{-m} \sum_{X=0}^{\infty} \frac{(me^{t})^{X}}{X^{\frac{1}{2}}}$$

 $= e^{-m} e^{me^{t}} e^{m(e^{t}-1)}$

(d) for the exponential distribution, N.) $M(t) = \frac{\lambda}{\lambda - t}.$ (3.56) the Probability density function Os an exponential dist. is $P(X) = \frac{5}{20} \frac{e^{-XX}}{15} \frac{15}{X60}$ $= \lambda e^{-\lambda \kappa} G(\kappa)$ where E is the heaviside Step function. $M(t) = \int dx Pcx) e^{tx}$ = $\lambda \int dx e^{-\lambda x} e^{tx} = \lambda \int dx e^{(t-\lambda)x}$ $= \lambda \int_{a} \frac{e^{(t-\lambda)\chi}}{t-\lambda} \int_{a}^{\infty}$ assuming t22



5. A model for rubber string is a one-dimensional chain of molecules, each with length a. Molecules are joined at their ends in a way, that the next one can go left or right with equal probability. If we denote n_+ the number going to the right and n_- the number going to the left, they satisfy the relations

$$n_+ + n_- = N, \quad X = a(n_+ - n_-)$$

where N is their total number and X is the length of the chain.

(a) what is the probability of having a given X, denoted by W(X)?

(b) Using Sterling's formula, calculate the entropy of the chain $S = k_B log(W)$

(c) Calculate the "entropic force"

$$F = -k_b T \frac{\partial S}{\partial X}$$

at small $x \ll Na$ and large $x \to Na$ length of the chain

Since the Chain caul Probability of gaing either direction $P(+) = \frac{1}{2} = P(-)$ Since in this case P(t) (un relasent a "Success" while P(-) is a "failure" in Bernaulli terms. Since there are N atoms (trials)

then the distribution of leusth is a Binomicl distribution. $P(n_{+}) = \binom{n}{n_{+}} (\frac{1}{2})^{n_{+}} (\frac{1}{2})^{n_{+}}$ $P(n_{+}, n_{-}) = \frac{n!}{n_{+}! n_{-}!} \left(\frac{1}{2}\right)^{n_{+}} \left(\frac{1}{2}\right)^{n_{-}}$ and N=n++na) $X \equiv a(n_t - n_-)$ $(X) = ((N_{*} - N_{-}))$ $= G(N_{+} - N_{-}) = G(N_{+}) - G(N_{-})$ Since Mt, n- have a Sym. 08 Podratilitz we will expect <N+7 = <N-7 => 2×7=0 Now let's Prove that

Gur generating function ir 36 is $M(t) = (Pe^{t} + 1 - P)^{n}$ and $\frac{\partial M}{\partial t^{K}} = \langle X^{K} \rangle$ Since $P = \frac{1}{2}, X = N_{+}$ $\langle n_{+} \rangle = \frac{\partial}{\partial t} \left(\frac{\partial}{\partial t} \right)^{\prime} \left(\frac{e^{t}}{t} + 1 \right)^{\prime} \right)_{t}$ $= (\frac{1}{2})^{n} n(e^{t}+1)^{n} \cdot e^{t}|_{t=0}$ $= (\frac{1}{2})^{N} N(2)^{N}$ = 12 < N-2 = 2N2 - 2N+2 = NX17 - M2

= N(2+2) - 2 $\mathcal{N}(0) = \frac{\partial \mathcal{M}}{\partial t^{\circ}} \bigg|_{t=0}$ $=) < N - 2 = N - \frac{1}{2} = \frac{1}{2}N$ th-s LX7= GLN+2 - GLN-2 $= \alpha \cdot \frac{N}{2} - \alpha \frac{N}{2} = 0$ Y L 5.) $X = a(n_{+} - n_{-}) = a(n_{+} - (N_{-} - n_{+}))$ $= G(2A_{+} - N)$ thus the # of ways to get a Particular X is dependent on the # of Ways to get N+ which is (n).

Putting
$$N_r$$
 in terms of X
we get
 $\frac{1}{2} \begin{pmatrix} X \\ a \end{pmatrix} = N_r$
 $= 2 \begin{pmatrix} X \\ a \end{pmatrix} = N_r$
 $= 2 \begin{pmatrix} N \\ (\frac{1}{2} \begin{pmatrix} X + N \end{pmatrix} \end{pmatrix} = (\frac{N!}{2 \begin{pmatrix} X + N \end{pmatrix}}) \begin{pmatrix} \frac{1}{2} \begin{pmatrix} N - \frac{X}{2} \end{pmatrix} \end{pmatrix}$

=>

$$S = k_{B} \left[h_{V} N! - h_{V} \left(\frac{1}{2} \left(N + \frac{x}{n} \right) \right)^{1} \right] - h_{V} \left(\frac{1}{2} \left(N - \frac{x}{n} \right)^{1} \right)^{1} \right]$$

$$S + er \lim_{n \to \infty} 5 \quad for mula \quad (S)$$

$$h_{V} \left(\frac{1}{2} \right)^{1} = J_{V} J_{V} - J_{V} - J_{V} \left(\frac{1}{2} \left(N + \frac{x}{n} \right) \right)^{1} \right]$$

$$=> S = k_{B} \int N J_{V} N - \frac{1}{2} \left(N + \frac{x}{n} \right) \int J_{V} \left(\frac{1}{2} \left(N + \frac{x}{n} \right) \right)^{1} - \frac{1}{2} \left(N - \frac{x}{n} \right) \int J_{V} \left(\frac{1}{2} \left(N - \frac{x}{n} \right) \right)^{1} - N + \frac{1}{2} \left(N - \frac{x}{n} \right)^{1} \right)$$

$$= \int \left(N - \frac{x}{n} \right) \int J_{V} \left(\frac{1}{2} \left(N - \frac{x}{n} \right) \right)^{1} - N + \frac{1}{2} \left(N + \frac{x}{n} \right)^{1} \right)$$

$$= \int \left(N - \frac{x}{n} \right) \int J_{V} \left(\frac{1}{2} \left(N - \frac{x}{n} \right) \right)^{1} - N + \frac{1}{2} \left(N + \frac{x}{n} \right)^{1} \right)$$

$$= \int \left(N \int J_{V} \left(\frac{1}{2} \left(N + \frac{x}{n} \right) \right) \int J_{V} \left(\frac{1}{2} \left(N - \frac{x}{n} \right) \right)^{1} \right)$$

$$(.) \quad F = -T \frac{\partial S}{\partial X}$$

$$\frac{\partial S}{\partial X} = \left[\frac{1}{2} + \frac{1}{2} \mu \left(\frac{1}{2} \left(N + \frac{X}{4} \right) \right) + \frac{1}{2} \mu \left(\frac{1}{2} \left(N - \frac{X}{4} \right) \right) + \frac{1}{2} \mu \left(\frac{1}{2} \left(N - \frac{X}{4} \right) \right) + \frac{1}{2} \mu \left(\frac{1}{2} \left(N - \frac{X}{4} \right) - \frac{1}{2} \frac{\left(N - \frac{X}{4} \right)}{\frac{1}{2} \left(N - \frac{X}{4} \right)} - \frac{\frac{1}{2} \frac{\left(N - \frac{X}{4} \right)}{\frac{1}{2} \left(N - \frac{X}{4} \right)} + \frac{1}{2} \left(\frac{N - \frac{X}{4}}{N + \frac{X}{4}} \right)$$

$$= \frac{K_0 T}{\alpha} \int_{\Omega} \left(\frac{N + \frac{X}{4}}{N - \frac{X}{4}} \right)$$

$$= \frac{K_0 T}{\alpha} \int_{\Omega} \left(\frac{N \alpha + X}{N \alpha - X} \right)$$

$$= \frac{K_0 T}{\alpha} \int_{\Omega} \left(\frac{1 + \frac{X}{4}}{N \alpha - X} \right)$$

 $\left(1 - \frac{1}{N_{n}} \right)$ if X<<Na=> × 2< ($20 \quad \frac{1+x_{a}}{1-x} \sim \left(1+\frac{x}{N_{a}}\right) \left(1+\frac{x}{N_{a}}+\frac{(x)^{2}}{(x_{a})^{2}+\cdots}\right)$ $= 1 + \frac{2x}{Na} + O\left(\frac{x}{Na}\right)^{2}$ Fr Kor hullting) $lu(1+\epsilon) = 0 + \epsilon - \frac{\epsilon}{2}$ $F_{N} \frac{k_{0} f(2x)}{N_{0}} = \frac{2 k_{0} T x}{N_{0}^{2}}$ if X->Na then Na > 1

 $\lim_{\varepsilon \to 1^{-}} \lim_{\varepsilon \to 1^{-}} \frac{1+\varepsilon}{1-\varepsilon} = \infty$

5- F-> ~ LS X->NG A possible reason is that our constraints assume plut the longest leasth 3 Na. IN order to go fast the lanst (cust h one needs to Crente a non-existent State.