HOMEWORK 1, THERMAL PHYSICS (PHY306)

(3.3) This question is about a *discrete* probability distribution known as the **Poisson distribution**. Let x be a discrete random variable that can take the values $0, 1, 2, \ldots$ A quantity x is said to be Poisson distributed if the probability $P(x)$ of obtaining x is

$$P(x) = \frac{e^{-m} m^x}{x!},$$

where m is a particular number (which we will show in part (b) of this exercise is the mean value of x).

(a) Show that $P(x)$ is a well-behaved probability distribution in the sense that

$$\sum_{x=0}^{\infty} P(x) = 1.$$

(Why is this condition important?)

(b) Show that the mean value of the probability distribution is $\langle x \rangle = \sum_{x=0}^{\infty} x P(x) = m$.

(c) The Poisson distribution is useful for describing very rare events, which occur independently and whose average rate does not change over the period of interest. Examples include birth defects measured per year, traffic accidents at a particular junction per year, numbers of typographical errors on a page, and the number of activations of a Geiger counter per minute. The first recorded example of a Poisson distribution, the one which in fact motivated Poisson, was connected with the rare event of someone being kicked to death by a horse in the Prussian army. The number of horse-kick deaths of Prussian military personnel was recorded for each of 10 corps in each of 20 years from 1875–1894 and the following data recorded:

Number of deaths per year, per corps	Observed frequency
0	109
1	65
2	22
3	3
4	1
≥ 5	0
Total	200

Calculate the mean number of deaths per year per corps. Compare the observed frequency with a calculated frequency assuming the number of deaths per year per corps are Poisson distributed with this mean.

FIGURE 1. Problem 1

Problem 2. (a) Using data from chapter 2, calculate how much energy is needed to make a cup of tea (.2 kg of water, from $T = 20^o C$ to $100^o C$).
(b) Assuming world energy production is $10^{13} W$, how much boiling water one can get in 1 year? Visualize it by assuming this water is making a cube of size L: how many meters such L will be? ($1\, m^3$ of water weights $10^3\, kg$.)

(3.8) This question introduces a rather efficient method for calculating the mean and variance of probability distributions. We define the **moment generating function** $M(t)$ for a random variable x by

$$M(t) = \langle \mathrm{e}^{tx} \rangle. \tag{3.51}$$

Show that this definition implies that

$$\langle x^n \rangle = M^{(n)}(0), \tag{3.52}$$

where $M^{(n)}(t) = \mathrm{d}^n M/\mathrm{d}t^n$ and further that the mean $\langle x \rangle = M^{(1)}(0)$ and the variance $\sigma_x = M^{(2)}(0) - [M^{(1)}(0)]^2$. Hence show that:

(a) for a single Bernoulli trial,

$$M(t) = p\mathrm{e}^t + 1 - p; \tag{3.53}$$

(b) for the binomial distribution,

$$M(t) = (p\mathrm{e}^t + 1 - p)^n; \tag{3.54}$$

(3.5) If θ is a continuous random variable which is uniformly distributed between 0 and π, write down an expression for $P(\theta)$. Hence find the value of the following averages:

(a) $\langle \theta \rangle$;
(b) $\langle \theta - \frac{\pi}{2} \rangle$;
(c) $\langle \theta^2 \rangle$;
(d) $\langle \theta^n \rangle$ (for the case $n \geq 0$);
(e) $\langle \cos\theta \rangle$;
(f) $\langle \sin\theta \rangle$;
(g) $\langle |\cos\theta| \rangle$;
(h) $\langle \cos^2\theta \rangle$;
(i) $\langle \sin^2\theta \rangle$;
(j) $\langle \cos^2\theta + \sin^2\theta \rangle$.

Check that your answers are what you expect.

FIGURE 2. Problems 3 and 4

Problem 5. A model for rubber string is a one-dimensional chain of molecules, each with length a. Molecules are joined at their ends in a way, that the next one can go left or right with equal probability. If we denote n_+ the number of molecules going to the right and n_- the number going to the left, they satisfy the relations

$$n_+ + n_- = N, \quad X = a(n_+ - n_-)$$

where N is their total number and X is the length of the chain.
(a) show that the probability of having a given length X, denoted by $P(X)$, is given by binomial distribution
(b) Using Sterling's formula, calculate the entropy of the chain $S(X) = k_B log(P(X))$
(c) Calculate the "entropic force"

$$F = -k_b T \frac{\partial S(X)}{\partial X}$$

for small chain $|x| \ll Na$ and near its maximal strenching $|x| \to Na$.

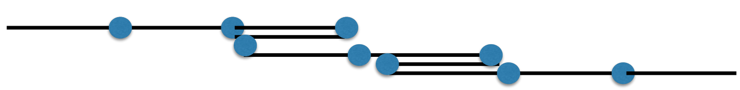

FIGURE 3. Sketch of the string for problem 5, with $N = 10, n_- = 2, n_+ = 8, X = 6a$.