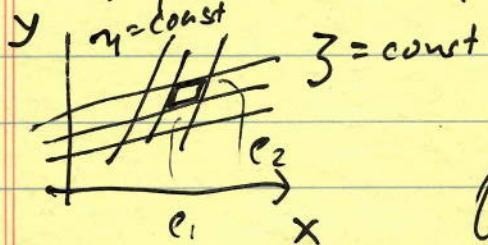


Additional material: Jacobians

let us consider 2d plane (x, y)

and families of lines

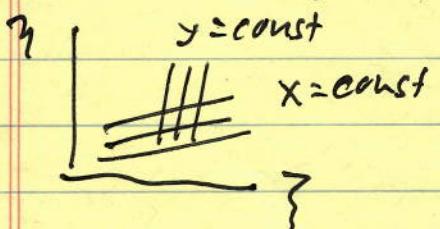


$$\begin{cases} \bar{z} = \bar{A}x + \bar{B}y \\ \bar{y} = \bar{C}x + \bar{D}y \end{cases}$$

One can solve linear eqn's
and obtain

\begin{cases} x = A\bar{z} + B\bar{y} \\ y = C\bar{z} + D\bar{y} \end{cases}

Matrix $M = \begin{pmatrix} \bar{A} & \bar{B} \\ \bar{C} & \bar{D} \end{pmatrix}$ and inverse $M^{-1} = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$



transition or map

$$(x, y) \xrightarrow{\quad} (\bar{z}, \bar{y})$$

Area transformation \rightarrow

Take element between $(\bar{z}, \bar{z} + \Delta\bar{z}, \bar{y}, \bar{y} + \Delta\bar{y})$
4 lines:

$$\Delta\Omega = \ell_1 \ell_2 |\sin(\alpha_1 - \alpha_2)| = \ell_1 \ell_2 |\sin \alpha_1, \cos \alpha_2 - \cos \alpha_1, \sin \alpha_2|$$

$$\text{Now: } \begin{cases} \ell_2 \sin \alpha_2 = \Delta y_{\bar{z}} & \ell_1 \sin \alpha_1 = \Delta y_{\bar{y}} \\ \ell_2 \cos \alpha_2 = \Delta x_{\bar{z}} & \ell_1 \cos \alpha_1 = \Delta x_{\bar{y}} \end{cases}$$

$$\text{and } \Delta\Omega = |\Delta x_{\bar{z}} \Delta y_{\bar{y}} - \Delta x_{\bar{y}} \Delta y_{\bar{z}}| = |AB - BC| \Delta\bar{z}\Delta\bar{y} = \det \begin{vmatrix} A & B \\ C & D \end{vmatrix} d\bar{z}d\bar{y}$$

General transformation

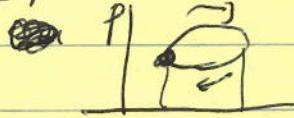
$$\begin{cases} \bar{z} = \bar{z}(x, y) & d\bar{z} = \frac{\partial \bar{z}}{\partial x} dx + \frac{\partial \bar{z}}{\partial y} dy \\ \bar{y} = \bar{y}(x, y) & dy = \dots \end{cases} \quad \text{matrix of derivatives}$$

$$d\bar{\Omega} = |D| d\bar{z}d\bar{y} : D \equiv \frac{\partial(x, y)}{\partial(\bar{z}, \bar{y})} = \begin{vmatrix} \frac{\partial x}{\partial \bar{z}} & \frac{\partial x}{\partial \bar{y}} \\ \frac{\partial y}{\partial \bar{z}} & \frac{\partial y}{\partial \bar{y}} \end{vmatrix}$$

Jacobian of inverse (old over new)
function

Thermodynamical variables:

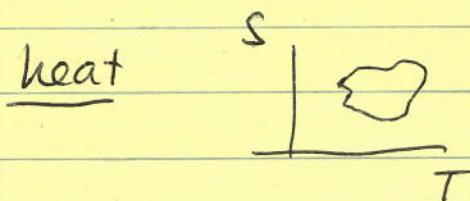
work $\int pdV$ is convenient to calculate on (P, V) plane



V

$\oint pdV$ cycle

Work = area of a cycle



$\oint TdS \rightarrow$ area of a cycle

Units better be calibrated in such a way that

Jacobian

$$\boxed{\frac{\partial(TS)}{\partial(PV)} = 1}$$

$$dU = dQ + dW$$

↑
internal
energy

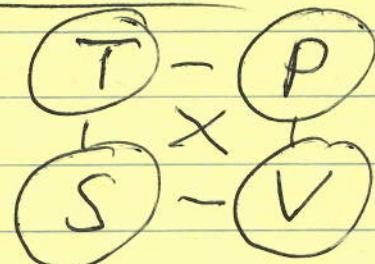
heat transferred to
the system
+ work done on the
system

$\oint dU = 0$ since U is function of state

thus $\oint dQ + \oint dW = 0$ energy conserved
the 1st law

4 variables (2 independent)

6 possible pairs



Although any curve can
be plotted at any of them, straight line parallel
to axes are the simplest:

T = const isothermal, P - isobaric,
V - isochoric ; S - adiabatic

Isothermal expansion

$$T = \text{const}$$

$$pV = \text{const}$$

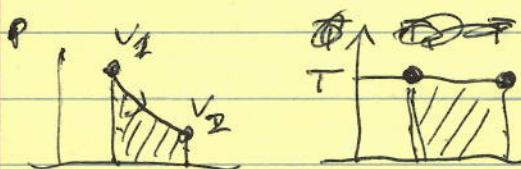
$$p = \frac{nRT}{V}$$

$$\Delta U = \Delta W + \Delta Q$$

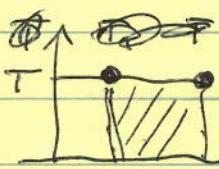
Because for ideal gas $T = f(T)$ only
So $\Delta Q = -\Delta W = -\int p dV$

argument
would not work,
for other systems!

$$= -RT \ln\left(\frac{V_2}{V_1}\right) \quad (12.8)$$



$$\Delta W = \int p dV$$



$$\int T dS = \Delta Q$$

But, because we already studied stat mech of ideal gas, we know $F = U - TS$ and ~~and~~

$$S = N k_B \log \left[\frac{V e^{S/2}}{N \left(\frac{2\pi h^2}{m k_B T} \right)^{3/2}} \right]$$

(Sackur-Tetrode)
(on isothermal line)

So, we can calculate $\Delta Q = T \Delta S$ directly.

$$T \Delta S = k_B T N \log\left(\frac{V_2}{V_1}\right), \text{ same result.}$$

Once again general overview
4 variables

$$\Delta U = T \Delta S - p \Delta V$$

$T \leftrightarrow P$
 $S \leftrightarrow V$
gives $Q \leftrightarrow W$

6 pairs \rightarrow 6 possible plots

Isotherm was special
 $dU = 0$, so $dW \neq dQ = 0$
at other lines $-dQ \neq dW$

A point at any plot means we know 2 variables and EOS allows to get this point at any other plot!

12.3. Adiabatic expansion (of an ideal gas)

* Argument in a Book: $\Delta Q = 0$ no heat

thus $\Delta U = \Delta W$ and we calc.
it on p-V plot

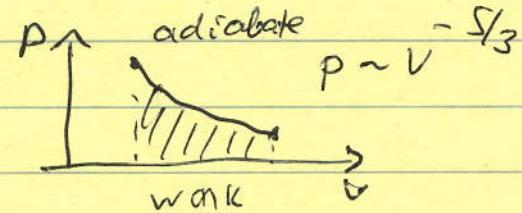
$$dU = C_V dT \stackrel{!}{=} dW = -pdV = -\frac{RT}{V} dV$$

$$\frac{dT}{T} = -\frac{R}{C_V} \frac{dV}{V} \rightarrow \ln \frac{T_2}{T_1} = -\frac{R}{C_V} \ln \frac{V_2}{V_1} \rightarrow$$

$$\rightarrow T V^{\frac{R}{C_V}} = \text{const} \quad | \text{ if } C_V = \frac{3}{2}R \text{ then}$$

or, changing $T \rightarrow \frac{PV}{\cancel{RT}}$

$$PV^{1+\frac{R}{C_V}} = PV \frac{C_P}{C_V} \rightarrow (5/3)$$



* Direct calculation $\Delta Q = T dS = 0$

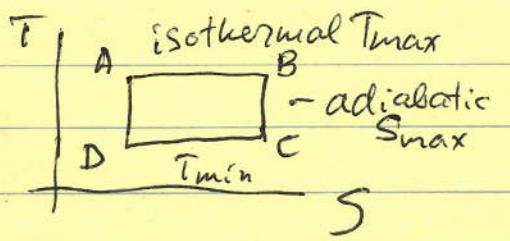
$$S = k_B N \log \left[\frac{V}{N} \frac{e^{5/2}}{\left(\frac{2\pi h^2}{m k_B T} \right)^{3/2}} \right] = T \underbrace{\left[\dots \right]}_{-S=\text{const.}} \quad | \text{ so } S = \text{const.} \quad \text{if } T dS = 0 \text{ no heat!}$$

$= \text{const}$

$$V \cdot T^{3/2} = \text{const} ; \quad T \rightarrow pV \quad \text{so} \quad p^{3/2} V^{7+3/2} = \text{const}$$

$$p V^{5/3} = \text{const}$$

The Carnot cycle



$$T_A = T_B \equiv T_{\max}$$

$$T_D = T_C \equiv T_{\min}$$

$$\Delta Q = (T_{\max} - T_{\min})(S_{\max} - S_{\min})$$

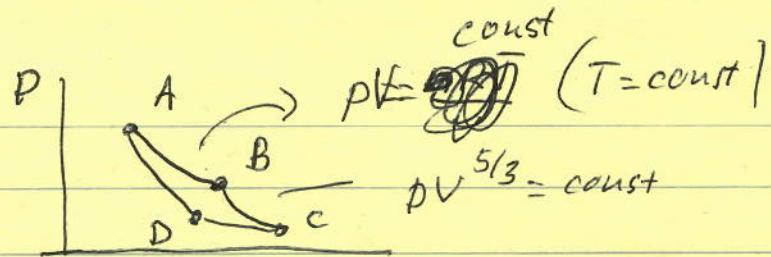
$$\Delta Q_{in} = T_{\max} \Delta S$$

$$\Delta Q_{out} = T_{\min} \Delta S$$

$$\frac{\Delta Q_{in}}{\Delta Q_{out}} = \frac{T_{\max}}{T_{\min}}$$

$$\eta = \frac{W}{Q_{in}} = \frac{Q_{in} - Q_{out}}{Q_{in}} = 1 - \frac{T_{\min}}{T_{\max}}$$

efficiency.



In a book calculation is in PV plot:

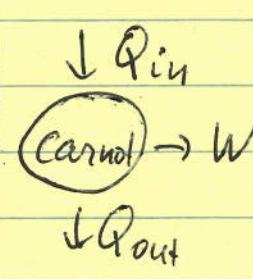
$$P_A V_A = P_B V_B$$

$$P_D V_D = P_C V_C$$

$$P_B V_B^{5/3} = P_C V_C^{5/3}$$

$$P_D V_D^{5/3} = P_A V_A^{5/3}$$

and ΔQ_{in} and ΔQ_{out} found
can be solved ~~but not~~
needed because of energy conservation

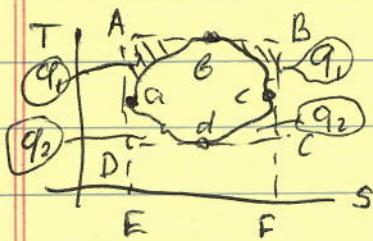


$$Q_{out} + W = Q_{in}$$

Carnot cycle is the most efficient one:

* Note first, that we have not used EOS $\rightarrow \eta$ the same for any matter
we only use energy conservation.

* Note that if $T_{\min} \rightarrow 0$ $\eta \rightarrow 1$ (no heat is lost)



Let us take arbitrary cycle (abcd)
 $\eta = \frac{Q_1}{Q_2}$ and make Carnot cycle surrounding

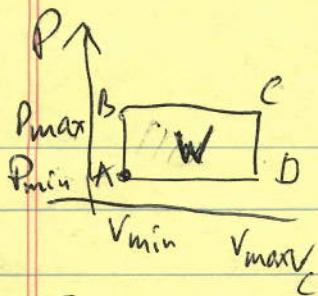
$$\text{area}(aAB) + \text{area}(BBC) \equiv q_1$$

$$\text{area}(cCd) + \text{area}(dDa) \equiv q_2$$

$$\eta_{\text{Carnot}} = \frac{\text{area}(ABFE) - \text{area}(DCFE)}{\text{area}(ABFE)}$$

$$= \eta_c - \frac{q_1}{\text{Area}(ABFE) - q_2} (1 - \eta_c) - \frac{(q_2)}{\text{Area}(ABFE) - q_2}$$

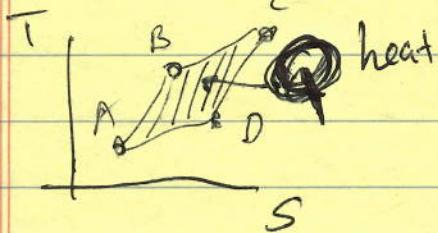
Both terms sign def.
 $\eta < \eta_c$



rectangular
cycle in (pV)
plane

$$W = (\Delta p)(\Delta V)$$

But what is heat?



$$\left\{ \begin{array}{l} p_A T_A = R T_A \\ \text{etc for } B, C, D \end{array} \right\} \text{ all } T \text{ are simple to get}$$

AB isochore $V = \text{const}$

$$(pV = Nk_B T)$$

$$S = Nk_B \log \left[\frac{V e^{5/2}}{N \left(\frac{2\pi h^2}{m k_B T} \right)^{3/2}} \right]$$

$$= \text{const} + Nk_B \left[\log V^{5/2} + \log p^{-3/2} \right]$$

So one can do S_A, S_B, S_C, S_D . (or any number of points)
on the line

$$W = \oint p dV$$

Any cycle can be mapped, e.g.

