Spectral shock waves in QCD

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45 Years of Nuclear Physics at Stony Brook
Tribute to Gerald E. Brown

Outline

- Strong-weak coupling and chiral transitions in QCD
- "Schematic" models
- Surfing the singularities
- Finite N as viscosity in the spectral flow – Burgers turbulence in QCD
- "What a beautiful catastrophe" - order-disorder phase transitions in QCD
- Summary

Two key operators in QCD

- Wilson loop operator $W(C) = \left\langle \mathrm{P} e^{i \oint_C A_\mu dx_\mu} \right\rangle_{QCD}$ controls weak-strong coupling transition in QCD
- Dirac operator $\langle \det(i\not{D}(A) + i\epsilon) \rangle_{QCD}$ is responsible for spontaneous breakdown of the chiral symmetry
  Banks-Casher relation for (Euclidean) QCD
  $< \bar{q}q >_{QCD} \sim \frac{\rho(0)}{V_4}$

Both transitions involve dramatic rearrangements of the eigenvalues, evolving as as a function of some exterior parameters (time, length of the box, area of the surface of the loop, temperature ...)
Our goal: In the spirit of Gerry's schematic models, let us find out "universality windows" where some simplified dynamics reflects the features of non-trivial theories

Two schematic models

I. Multiplicative schematic model

- $W = U_1 U_2 ... U_k$
- All U are unitary, $N \times N$
- Dynamics from randomness
- Evolution starts at $\mathbf{1}_N$
- Eigenvalues on the unit circle
- Analog of multiplicative Brownian walk $x_{k+1} - x_k = x_k \cdot dB(t)$

II. Additive schematic model

- $D = H_1^\chi + H_2^\chi + ... + H_k^\chi$
- H_i^χ are chiral hermitian $[H_i^\chi, \gamma_5]_+ = 0$
  i.e. $H^\chi = \mathrm{offdiag}(H_{N\times M}, H_{M\times N}^\dagger)$ and $\gamma_5 = \mathrm{diag}(\mathbf{1}_N, -\mathbf{1}_M)$
- Evolution respects χ-symmetry
- Eigenvalues on real axis, $\nu = M - N$ mimic topological zero modes
- Analog of additive Brownian walk $x_{k+1} - x_k = dB(t)$

Spectral flow of random matrices

- Price/prize from switching to eigenvalues from matrix element: Jacobian $J = \prod_{i<j}(\lambda_i - \lambda_j)^2$
- Jacobian triggers $d = 2$ Coulombic, repulsive interactions between eigenvalues $J = e^{\sum \ln|\lambda_i - \lambda_j|}$; [Dyson;1962]
- Brownian walk of eigenvalues gets a drift from electric force $d\lambda_i = dB_i(t) + \sum_j \frac{dt}{\lambda_i - \lambda_j}$
- Evolution equation will develop non-linearities
- All nontrivial correlations in the spectral observables reflect this interaction

Complex Burgers equation

- From Langevin equation to Smoluchowski-Fokker-Planck equation for $P(\lambda_1, ..., \lambda_N, t) d\lambda_1 ... d\lambda_N$
- Integrating over all eigenvalues except one, rescaling time $t = N\tau$ gives equations for average spectral density $\rho(\lambda, \tau)$
- Resolvent $G_X(z) = \frac{1}{N} \left\langle \mathrm{Tr} \frac{1}{z-X} \right\rangle$, imaginary part gives the spectral function
- In the large N limit, resolvent fulfills inviscid complex Burgers (Euler) equation $\partial_\tau G(z,\tau) + G(z,\tau)\partial_z G(z,\tau) = 0$

Real Burgers equation

- $\partial_t f(x,t) + f(x,t)\partial_x f(x,t) = \mu \partial_{xx} f(x,t)$
  $f(x,t)$ is the velocity field at time t and position x of the fluid with viscosity μ.
- One-dimensional toy model for turbulence [Burgers 1939]
- But, equation turned out to be exactly integrable [Hopf 1950],[Cole 1951]
  If $f(x,t) = -2\mu\partial_x \ln d(x,t)$, then
  $\partial_t d(x,t) = \mu\partial_{xx} d(x,t)$ (diffusion equation), so general solution comes from Cole-Hopf transformation where
  $d(x,t) = \frac{1}{\sqrt{4\pi\mu t}} \int_{-\infty}^{+\infty} e^{-\frac{(x-x')^2}{4\mu t} - \frac{1}{2\mu}\int_0^{x'} f(x'',0)dx''} dx'$

Inviscid real Burgers equation

- $\partial_t f(x,t) + f(x,t)\partial_x f(x,t) = 0$
  and $f(x,0) = f_0(x)$.
- Method of characteristics: If $x(t)$ is the solution of ODE $dx(t)/dt = f(x(t),t)$, then $F(t) \equiv f(x(t),t)$ is constant in time along characteristic curve on the (x,t) plane
- Then $dx/dt = F$ and $dF/dt = 0$ lead to $x(t) = x(0) + tF(0)$ and $F(t) = F(0)$
- Defining $\xi \equiv x(0)$ we get
  $f(x,t) = f(\xi,0) = f_0(\xi) = f_0(x - tf(x,t))$, i.e. implicit relation determining the solution of the Burgers equation.
- When $d\xi/dx = \infty$, we get the shock wave.

Inviscid real Burgers equation

- In the case of inviscid Burgers equation, characteristics are straight lines, but with different slopes (velocity depends on the position)
- Characteristics method fails when lines cross (shock wave)
- Finite viscosity (or diffusive constant) smoothens the shock
- Inviscid limit of viscid Burgers equation is highly non-trivial, universal forerunner announces the shock

Do we have Burgulence in QCD?

- Q: Where are the shocks?
- A: Endpoints of the spectra of Wilson/Dirac operators correspond to shocks
- Q: What plays the role of viscosity?
- A: Viscosity $|\mu| = \frac{1}{2N}$
- Q: Is the analogy exact?
- A: Modulo technicalities (complex analysis instead of real analysis), **yes!**

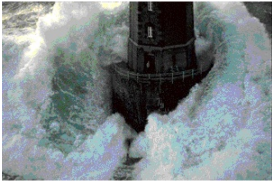
Surfing the shock waves

- Tracing the singularities of the flow allows to understand the pattern of the evolution of the complex system without explicit solutions of the complicated hydrodynamic equations...
- Zooming at singularities allows to infer the universal scaling (critical) exponents, since viscous equations are exact for arbitrary number of colors.
- Positive viscosity smoothens the shocks, negative viscosity is roughening the shock: origin of the universal oscillations anticipating the shock.

Visualization of the schematic model I:

- Collision of two shock waves, propagating along the unit circle

Gapped phase
$\tau < \tau^*$



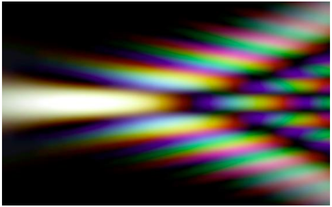
Closure of the gap
$\tau = \tau^*$

Gappless phase
$\tau > \tau^*$

Photos by Jean Guichard (La Jument lighthouse, Brittany)

Universal scaling visualization - "classical" analogy

Caustics, illustration from Henrik Wann Jensen



Fold and cusp fringes, illustrations by Sir Michael Berry

Morphology of singularity (Thom, Berry, Howls)

GEOMETRIC OPTICS
(wavelength $\lambda = 0$)

- trajectories: rays of light
- intensity surface: caustic

WAVE OPTICS ($\lambda \to 0$)

$N \to \infty$ Yang-Mills
($\mu = \frac{1}{2N} = 0$)

- trajectories: characteristics
- singularities of spectral flow

FINITE N YM (viscosity $\mu \to 0$)

Universal Scaling, Arnold (α) and Berry (σ) indices

"Wave packet" scaling
(interference regime)

- $\Psi = \frac{C}{\lambda^\alpha}\Psi(\frac{x}{\lambda^{\sigma_x}}, \frac{y}{\lambda^{\sigma_y}})$
- fold $\alpha = \frac{1}{6}$ $\sigma = \frac{2}{3}$ Airy
- cusp $\alpha = \frac{1}{4}$ $\sigma_x = \frac{1}{2}$ $\sigma_y = \frac{3}{4}$ Pearcey

Yang-Lee zeroes scaling with N
(for $N \to \infty$)

- YL zeroes of Wilson loop
- $N^{2/3}$ scaling at the edge
- $N^{1/2}$ and $N^{3/4}$ scaling at the closure of the gap

Large N Yang-Mills cont.

- Loop equations for $d = 2$ Yang-Mills theory Durhuus, Olesen, Migdal, Makeenko, Kostov, Matytsin, Gross, Gopakumar, Douglas, Rossi, Kazakov, Voiculescu, Pandey, Shukla, Janik, Wieczorek, Neuberger
- Burgers turbulence[Blaizot,MAN;2008]
- Universal preshock - expansion at the singularity for finite N, [Blaizot,MAN;2008]; [Neuberger;2008]
- Lattice confirmation: Narayanan, Neuberger, Lohmeyer;2006-2012

Lattice verification

Eigenvalues of regularized Wilson square loop of size 0.54fm for $N = 29$ versus eigenvalues of schematic model [Lohmayer and Neuberger; 2012].

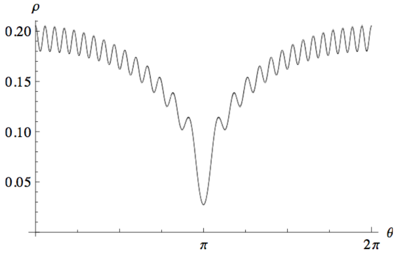
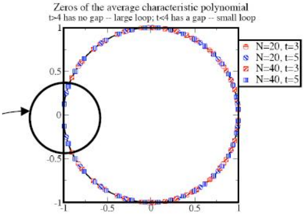


FIG. 1. Histogram of eigenvalue angles for a Wilson loop.

Wilson loops in large N Yang-Mills theories (time $\equiv$ area)

Numerical studies on the lattice (Narayanan and Neuberger, 2006-2007)

- $W(c) = \left\langle P \exp(i \oint A_\mu dx^\mu) \right\rangle_{YM}$
- $Q_N(z, \mathcal{A}) \equiv \langle det(z - W(\mathcal{A})) \rangle$
- Double scaling limit...
- $z = -e^{-y}$
  $y = \frac{2}{12^{1/4} N^{3/4}} \xi$
  $\mathcal{A}^{-1} = \mathcal{A}^{*-1} + \frac{\alpha}{4\sqrt{3}} \frac{1}{N^{1/2}}$
- $Q_N(z, \mathcal{A}) \to$
  $\lim_{N\to\infty} \left(\frac{4N}{3}\right)^{1/4} Z_N(\Theta, \mathcal{A}) =$
  $= \int_{-\infty}^{+\infty} du e^{-u^4 - \alpha u^2 + \xi u}$



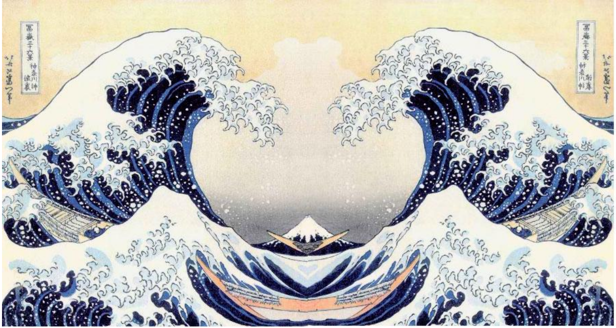
universality!

Closing of the gap is universal in $d = 2, 3, 4$

SBχS in QCD comes from merging of chiral shock waves

- Bulk properties of instanton vacuum agree with Dyson kernel universality [MAN, Verbaarschot, Zahed; 1988]
- At the closure of the gap, universal spectral oscillations are governed by Bessel type functions [Shuryak, Verbaarschot, Zahed;1993]
- Universal spectral window is limited by "Thouless scale" $\Lambda/\Delta = F_\pi^2\sqrt{V_4} \sim N\sqrt{V_4}$ [Verbarschot et al, Janik et al; 1998]
- Large windows and chiral shock collisions visible either in large N limit but fixed volume [Neuberger, Narayanan; 2004,2010] or in fixed $N = 3$ but large volume limit (Lattice community mainstream)

Breaking the chiral symmetry



Colliding Great Waves at Chiral Gap (collage based on Hokusai woodcut)

Chiral GUE - three types of shocks ([Blaizot, MAN, Warchoł;2013])

- In large N, M limit, but $\nu = M - N$ fixed, we again recover inviscid Burgers equation for relevant Green's function $\partial_\tau g(w,\tau) + g(w,\tau)\partial_w g(w,\tau) = 0$
- For general boundary condition we always get three types of shock waves:
- If we define $w - w^* \equiv p$ we get three types of scalings
  1. $p \to N^{-2/3} s$ for $\tau < 1$
  2. $p \to N^{-3/4} s$ for $\tau = 1$
  3. $p \to N^{-1} s$ for $\tau > 1$

Chiral GUE - Bessel and Bessoid "heralds" of shocks

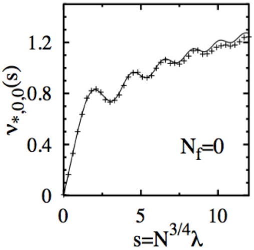
Solutions in the vicinity of shocks

1. For $\tau < 1$, Airy edge, where $Ai(-\sqrt{2\dot{g}_0 s})$,
2. For $\tau > 1$, Bessel edge, where $s^{-\nu/2} J_\nu(\pi\rho(0)\sqrt{s})$
3. For $\tau = 1$, generalized Bessoid $\int_0^\infty y^{\nu+1} e^{-y^4/2-y^2 r} J_\nu(2my)dy$ where variables $m = -is, r$ scale with N like $N^{3/4}$, $N^{1/2}$, respectively.

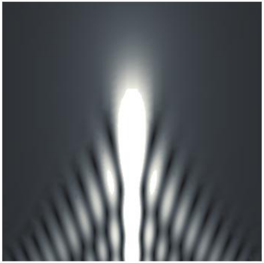
Note that Bessoid cusp in chiral GUE superimposes the Pearcey cusp known from Wilson loop, since chiral symmetry imposes azimuthal symmetry in the "Burgulence".

Bessoid visualizations

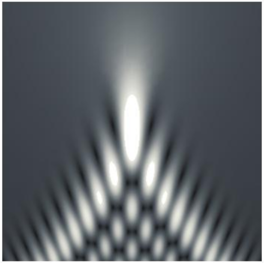
Similar pattern in optics (chiral diffraction) exists [Berry, Jenkins;2006], but was not yet measured.



Janik et al 1998



Modulus of Bessoid cusp



Modulus of Pearcey cusp

Conclusions

- New insight for order-disorder QCD transitions (e.g. Durhuus-Olesen transition, chiral symmetry breakdown)
- Generalizations for many flavors, topological charges, external parameters etc
- Generalizations of diffusion scenario for non-hermitian operators ([Gudowska-Nowak et al; 2003]), relevant e.g. for supersymmetric Wilson loops or finite chemical potential Dirac operators.
- Eye-opener/testbed for lattice calculations