# Real-Time Spectral Functions from the Functional Renormalization Group

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to Gerry Brown

an inspring physicist and a great human being

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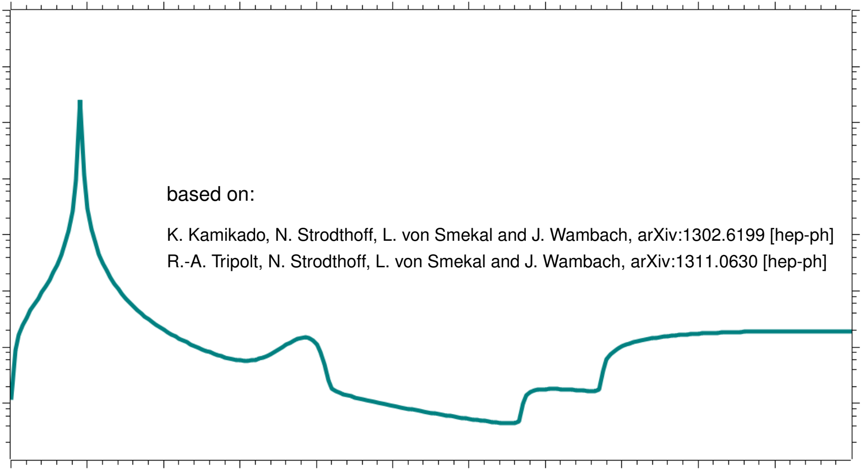

based on:

K. Kamikado, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1302.6199 [hep-ph]

R.-A. Tripolt, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1311.0630 [hep-ph]

# Outline

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- motivation
- theoretical setup:
  - Functional Renormalization Group (FRG)
  - analytic continuation from imaginary to real time
- results for the O(4) model at $T = 0$
- results for the quark-meson model at $T > 0$ and $\mu > 0$
- summary and outlook

# Motivation

- FRG well suited for the description of critial phenomena $\rightarrow$ phases of QCD matter
- formulated in imaginary time $\rightarrow$ no spectral information (Minkowski space-time)
- problem for real-time observables: analytic continuation

  - e.g. Lattice QCD: numerical reconstruction of real-time correlation functions (MEM), difficult if Euclidean data is not dense and precise enough

- we use non-perturbative FRG flow equations for two-point correlation functions and perform the analytic continuation on the level of the flow equations

# Functional Renormalization Group

**a primer**

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partition function: (scalar field $\phi(x)$)

$$Z[j] = e^{W[j]} = \int [\mathcal{D}\phi] \; e^{-S[\phi] + \int d^4x \; \phi(x) j(x)}$$

generating functional:

$$\left. \frac{\delta W[j]}{\delta j(x)} \right|_{j=0} = \frac{1}{Z[0]} \int [\mathcal{D}\phi] \phi e^{-S[\phi]} = \langle \phi(x) \rangle \equiv \varphi(x)$$

two-point correlation function: (Euclidean)

$$\left. \frac{\delta^2 W[j]}{\delta j(x) \delta j(y)} \right|_{j=0} = \langle \phi(x)\phi(y) \rangle - \langle \phi(x) \rangle \langle \phi(y) \rangle \equiv G(x,y)$$

effective action: ((Legendre transform of W)

$$\Gamma[\varphi] = -W[j] + \int d^4x \; \varphi(x) j(x)$$

stationarity condition: and thermodynamic potential:

$$\left. \frac{\delta \Gamma[\varphi]}{\delta \varphi} \right|_{\varphi=\varphi_0} = 0; \quad \rightarrow \quad \Omega(T) = \frac{T}{V} \Gamma[\varphi_0]$$

# Functional Renormalization Group

## Wilsonian coarse graining

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at given resolution scale $k$ split $\phi$ into low- and high-frequency modes:

$$\phi(x) = \phi_{q\le k}(x) + \phi_{q>k}(x)$$

$$\rightarrow \quad Z[j] = \int [\mathcal{D}\phi]_{q\le k} \underbrace{\int [\mathcal{D}\phi]_{q>k}\, e^{-S[\phi]+\int d^4x\ \phi j}}_{=Z_k[j]}\ ; \qquad \lim_{k\to 0} Z_k[j] = Z[j]$$

regulator $R_k(q)$:

$$\lim_{k\to 0} R_k(q) = 0$$

$$\lim_{k\to \Lambda} R_k(q) = \infty$$

$$Z_k[j] = \int [\mathcal{D}\phi]\, e^{-S[\phi]-\Delta S_k[\phi]+\int d^4x\ \phi j}$$

$$\Delta S_k[\phi] = \underbrace{\frac{1}{2}\int \frac{d^4q}{(2\pi)^4}\ \phi(-q)R_k(q)\phi(q)}_{\text{acts like a mass term } m_k}$$

effective action:

$$\rightarrow \quad \Gamma_k[\varphi] = -\ln Z_k[j] + \int d^4x\ \varphi(x)j(x) - \Delta S_k[\varphi]$$

$\Gamma_k$ interpolates between $k = \Lambda$ (no fluct.) and $k = 0$ (full quantum action)

$$\lim_{k\to\Lambda} \Gamma_k[\varphi] = S[\varphi]; \qquad \lim_{k\to 0} \Gamma_k[\varphi] = \Gamma[\varphi]$$

# Functional Renormalization Group

**flow equation** **C. Wetterich (1993)**

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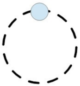
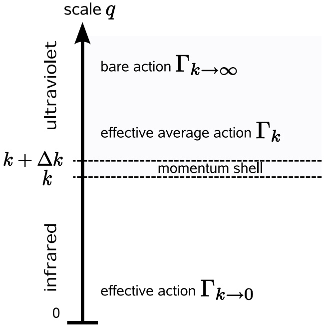
## flow equation for the effective action:

$$\partial_k \Gamma_k[\varphi] = \frac{1}{2} \mathrm{Tr} \left( \partial_k R_k \left[ \Gamma_k^{(2)} + R_k \right]^{-1} \right)$$

$$\Gamma_k^{(2)}(q) = \frac{\delta^2 \Gamma_k[\varphi]}{\delta\varphi(-q)\delta\varphi(q)}$$

[C. Wetterich, Phys. Lett. B301 (1993) 90]

$$\partial_k \Gamma_k = \frac{1}{2}$$ 



# O(4) Model

## flow equation

effective action: ('Local Potential Approximation')

$$\varphi = (\varphi_1, \dots, \varphi_4) = (\vec{\pi}, \sigma)$$

$$\Gamma_k[\varphi] = \int d^4x \left\{ \frac{1}{2}(\partial_\mu \varphi)^2 + U_k(\varphi^2) - c\sigma \right\}; \quad \varphi^2 = \varphi_i \varphi^i = \sigma^2 + \vec{\pi}^2$$

$$\Gamma^{(2)}_{k,ij}(q) = \Gamma^{(2)}_{k,\pi}(q) \left\{ \delta_{ij} - \frac{\varphi_i \varphi_j}{\varphi^2} \right\} + \Gamma^{(2)}_{k,\sigma}(q) \frac{\varphi_i \varphi_j}{\varphi^2}$$

flow equation for the effective potential:

$$\partial_k U_K = I_\sigma + 3 I_\pi; \quad I_i = \frac{1}{2} \mathrm{Tr}_q \left( \partial_k R_k(q) \left[ \Gamma^{(2)}_{k,i}(q) + R_k(q) \right]^{-1} \right)$$

for

$$R_k(q) = (k^2 - q^2)\Theta(k^2 - q^2)$$

one gets

$$I_i = \frac{k^4}{3\pi^2} \frac{1}{2E_i}, \qquad \text{with} \qquad E_\pi = \sqrt{k^2 + 2U'}; \quad E_\sigma = \sqrt{k^2 + 2U' + 4U''\varphi^2}; \quad U' = \left. \frac{\partial U}{\partial \varphi} \right|_{\varphi = \varphi_0} \quad etc.$$

# O(4) Model

## flow equations for 2-point functions

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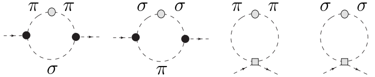
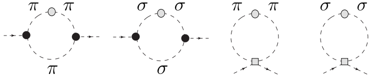
taking two functional derivatives of the flow equation for $\Gamma_k$ yields

$$\frac{\partial \Gamma^{(2)}_{\pi k}}{\partial k} \sim$$




$$\frac{\partial \Gamma^{(2)}_{\sigma k}}{\partial k} \sim$$




approximation:

$$\Gamma^{(3)}_{ijm} \to \frac{\delta^3 \Gamma_k}{\delta\varphi^m \delta\varphi^j \delta\varphi^i}; \qquad \Gamma^{(4)}_{ijmn} \to \frac{\delta^4 \Gamma_k}{\delta\varphi^n \delta\varphi^m \delta\varphi^j \delta\varphi^i}$$

ensures that truncation is consistent with effective action

$$\begin{aligned} \partial_k \Gamma^{(2)}_{k,\pi}(p=0) &= 2\partial_k U'_k \\ \partial_k \Gamma^{(2)}_{k,\sigma}(p=0) &= 2\partial_k U'_k + 4\partial_k U'' \varphi^2 \end{aligned}$$

and yields Nambu-Golstone boson in the chiral limit

# O(4) Model

## analytic continuation

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- first solve flow equation for the effective potential, $\partial_k U_k$
- substitute $p_0$ by continuous real frequency $\omega$

$$\Gamma^{(2),R}_{k,j}(\omega) = \lim_{\epsilon \to 0} \Gamma^{(2),R}_{k,j}(p_0 = -i(\omega + i\epsilon), \vec{p} = 0); \quad \text{for} \quad j = \pi, \sigma$$

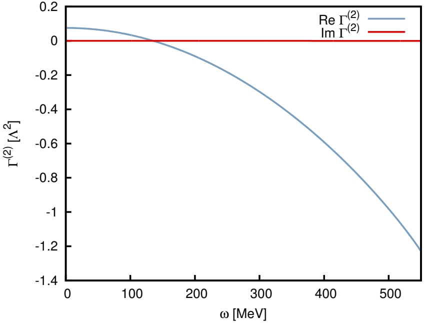
- then solve flow equations Re $\partial_k \Gamma^{(2),R}_k$, Im $\partial_k \Gamma^{(2),R}_k$ at global minimum of $U_{k \to 0}$
- finally, spectral functions are given by discontinuity of the propagators, i.e.

$$\rho_j(\omega) = -\frac{1}{\pi} \frac{\mathrm{Im}\,\Gamma^{(2),R}_j(\omega)}{\left(\mathrm{Re}\,\Gamma^{(2),R}_j(\omega)\right)^2 + \left(\mathrm{Im}\,\Gamma^{(2),R}_j(\omega)\right)^2}; \qquad \Gamma^{(2),R}_j(\omega) = \lim_{k \to 0} \Gamma^{(2),R}_{k,j}(\omega)$$

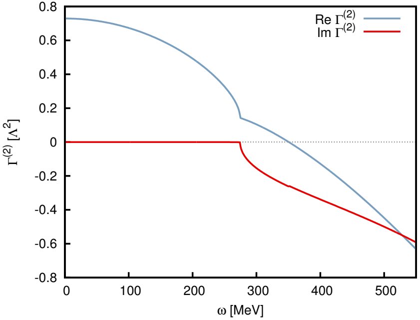
# Results for O(4) Model in vacuum

**Re $\Gamma^{(2),R}(\omega)$ and Im $\Gamma^{(2),R}(\omega)$**

pion:




sigma:



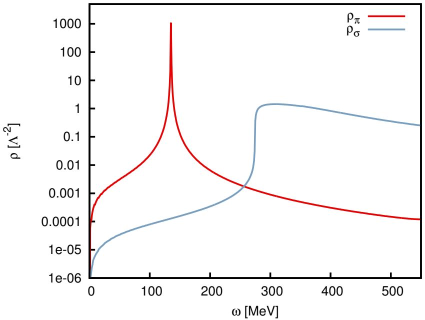

[K. Kamikado, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1302.6199 [hep-ph]]

# Results for O(4) Model in vacuum

## spectral functions

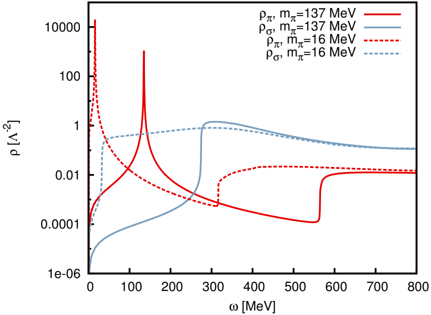
sigma and pion spectral functions

physical pion mass




sigma and pion spectral functions

chiral limit




[K. Kamikado, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1302.6199 [hep-ph]]

# Spectral Functions in a Thermal Medium

## Quark-Meson Model

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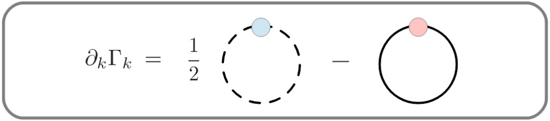
effective action:

$$\Gamma_k[\bar{\psi}, \psi, \varphi] = \int d^4x \left\{ \bar{\psi} \left( \partial\!\!\!/ + h(\sigma + i\vec{\tau} \cdot \vec{\pi}\gamma_5) - \mu\gamma_0 \right) \psi + \frac{1}{2}(\partial_\mu \varphi)^2 + U_k(\varphi^2) - c\sigma \right\}$$

- effective low-energy model for QCD with two flavors
- describes spontaneous and explicit chiral symmetry breaking
- flow equation for the effective action:



# Spectral Functions in the Medium

## phase diagram, masses and order parameter

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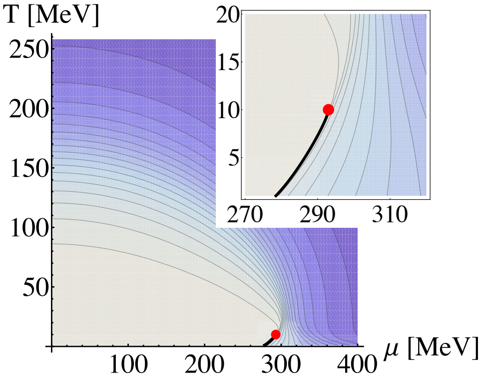
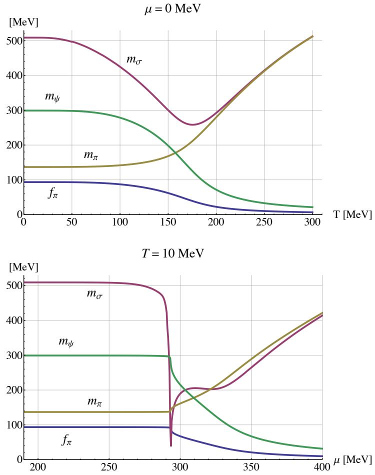


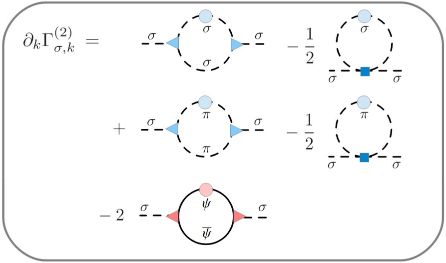



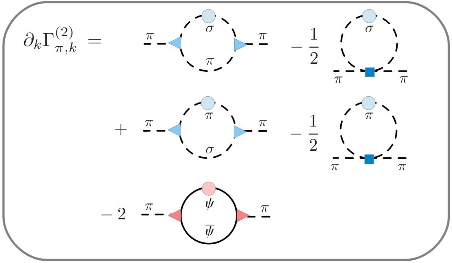

[R.-A. Tripolt, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1311.0630 [hep-ph]]

# Spectral Functions in the Medium

## flow of the two-point functions



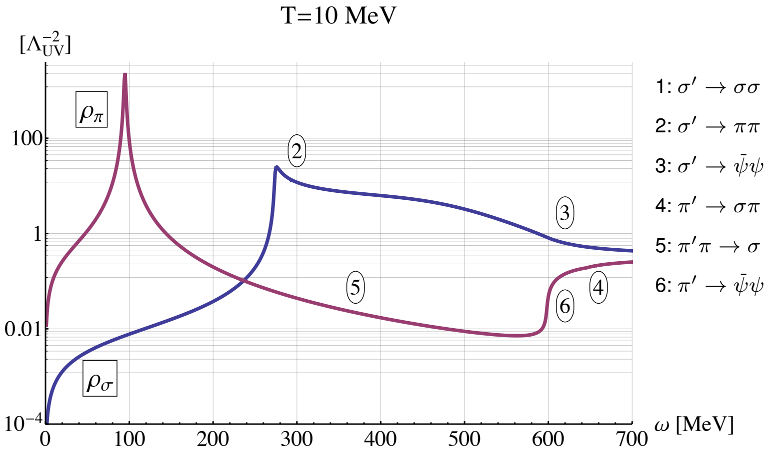



- quark-meson vertices given by $\Gamma^{(2,1)}_{\bar{\psi}\psi\sigma} = h$, $\Gamma^{(2,1)}_{\bar{\psi}\psi\vec{\pi}} = ih\gamma^5\vec{\tau}$
- meson vertices from scale-dependent effective potential: $\Gamma^{(0,3)}_{\phi_i\phi_j\phi_m}$, $\Gamma^{(0,4)}_{\phi_i\phi_j\phi_m\phi_n}$

# Results for Quark-Meson Model

## spectral functions at $\mu$ = 0

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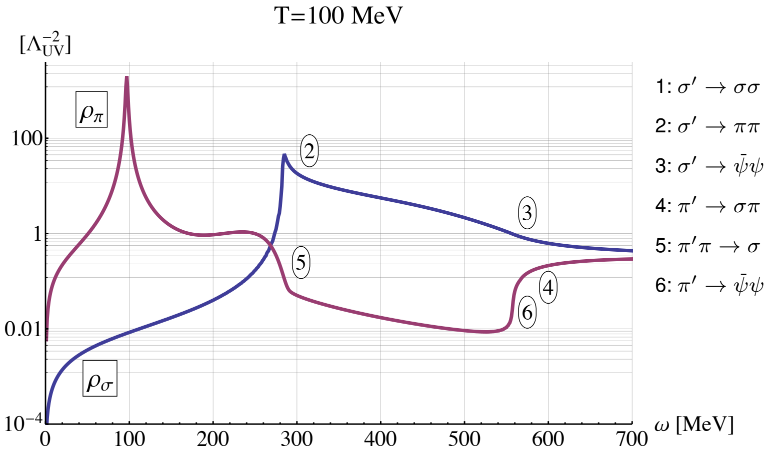
# Results for Quark-Meson Model

**spectral functions at $\mu$ = 0**

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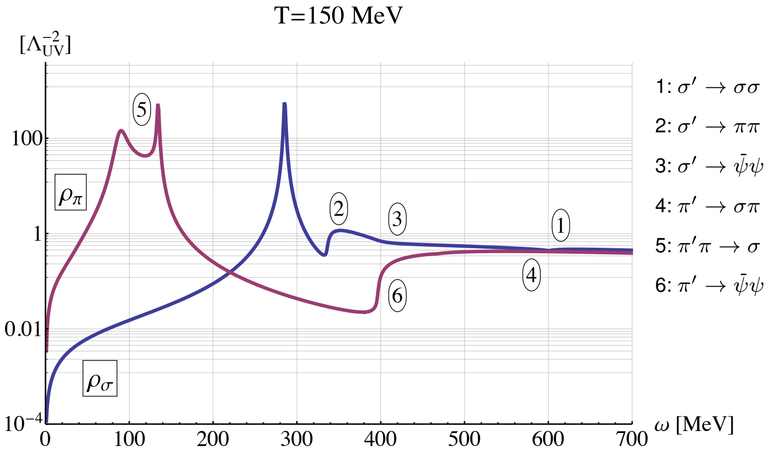


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# Results for Quark-Meson Model

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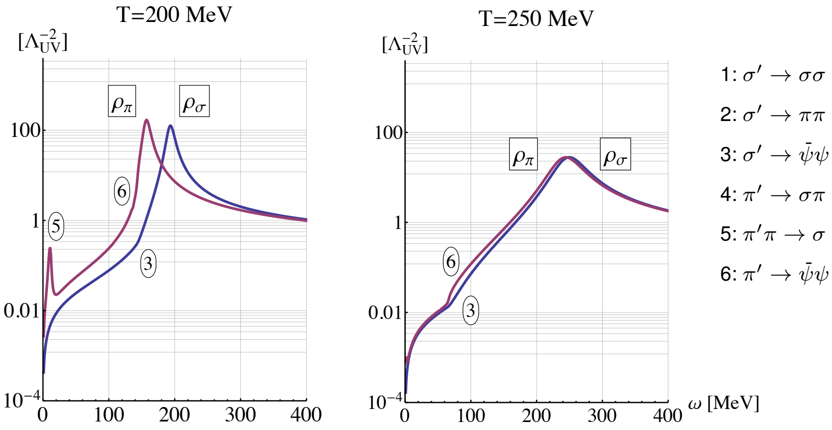


[R.-A. Tripolt, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1311.0630 [hep-ph]]

# Temperature Evolution

## animation

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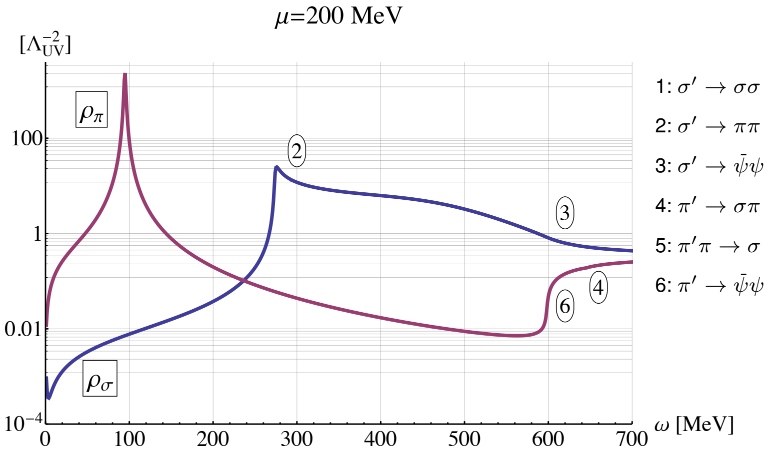
# Results for Quark-Meson Model

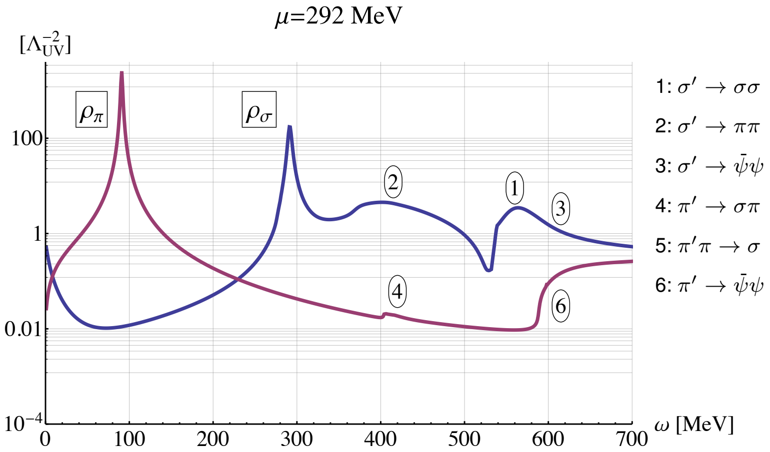
## spectral functions at finite $\mu$




[R.-A. Tripolt, N. Strodthoff, L. von Smekal and J. Wambach, arXiv:1311.0630 [hep-ph]]

# Results for Quark-Meson Model

## spectral functions at finite $\mu$

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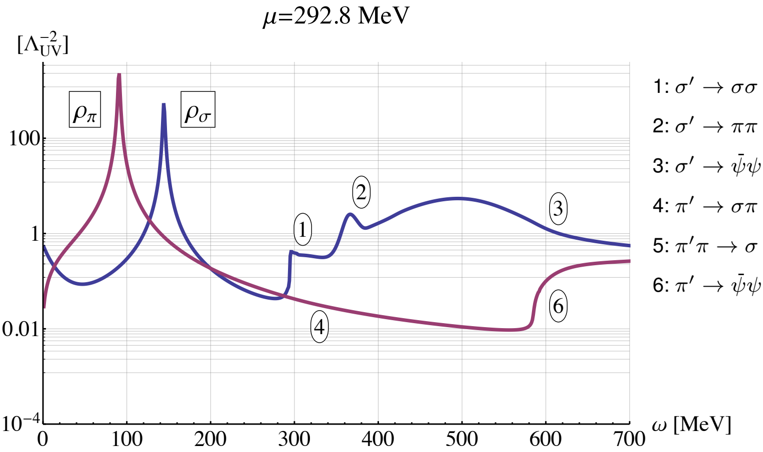
# Results for Quark-Meson Model

## spectral functions at finite $\mu$

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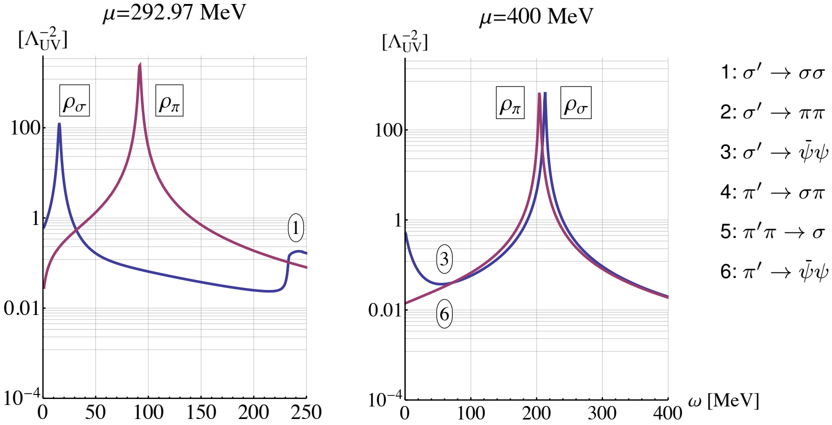


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# Summary and Outlook

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- presented a tractable method to obtain hadronic spectral functions at finite $T$ and $\mu$ from the FRG
- involves analytic continuation from imaginary to real frequencies at level of flow equations for 2-point functions
- results reveal complicated structure for in-medium spectral functions
- inclusion of finite external spatial momenta will allow for calculation of transport coefficients like shear viscosity