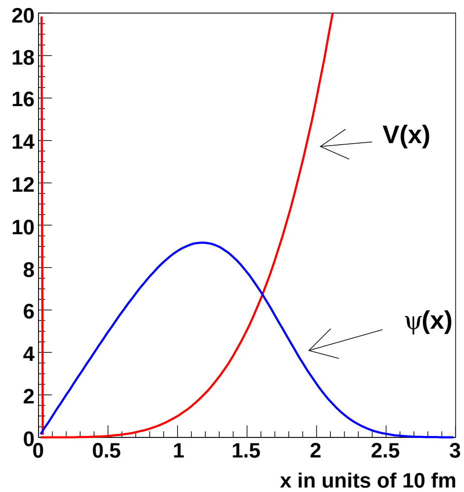
A schematic plot of the ground state wave function of a proton inside a nucleus is shown below. The units on the y axis are abitrary since the wave functions are not normalized. The units on the x axis are: $1unit = 10$Fm where $1\text{Fm} = 10^{-15}m$. (1 Fm known as a femto-meter or a fermi.)



a. Make an order or magnitude estimate for the Kinetic, Potential , and total energies based on this figure.

b. Treat the nucleus as 1D box of size L, determine the L which would give the same energy as you estimated in part a.

c. Suppose the proton were in this 1D box and decayed from the first excited state down to the ground state, what would the energy be of the emitted photon.

Solution

A. From the plot we see that the size of the wave functions is $L \sim 10\,\text{Fm}$. We then estimate that the momentum is $p \sim \hbar/L$, so the kinetic energy is

$$KE \sim \frac{\hbar^2}{2ML^2}$$

and for a bound state

$$PE \sim KE$$

Substituting Numbers

$$KE \sim \frac{(\hbar c)^2}{2M_p c^2} \tag{1}$$

with $\hbar c = 197\,\text{MeV Fm}$ (like $\hbar c = 197\,\text{eV nm}$ and $M_p c^2 = 938\,\text{MeV}$ we have

$$KE \sim PE \sim 0.2\,\text{MeV} \tag{2}$$

And

$$\text{Total E} \sim \text{KE} + \text{PE} \sim 0.4\,\text{MeV}$$

B. For a box

$$E = \frac{\hbar^2 \pi^2}{2M_p L^2}$$

So with $E = 0.4\,MeV$ and the same values as above

$$\begin{aligned} L &= \pi \sqrt{\frac{(\hbar c)^2}{2M_p c^2 E}} \\ &= 22.59\,\text{Fm} \end{aligned}$$

C. The energies are

$$E = \frac{\hbar^2 \pi^2}{2ML^2}\, n^2 \tag{3}$$

Then for the transition from $n = 2$ to $n = 1$ we have

$$\begin{aligned} E_{21} &= \frac{\hbar^2 \pi^2}{2ML^2}\,(2^2 - 1^2) \\ &= 3\,\frac{\hbar^2 \pi^2}{2ML^2} \\ &= 3\,(0.4MeV) \\ &= 1.2\,MeV \end{aligned}$$

Consider modelling the vibrations of HCl, made up of hydrogen and clorine atoms in an ionic bond. Make a schematic model of these vibration by considering the clorine to be very heavy (it is 35 times heavier than the hydrogen atom). Then the force between the hydrogen and the clorine is approximately proportional to the displacement x of the hydrogen from its equilibrium position. This means that the potential is like a spring, $\frac{1}{2}kx^2$. The energy of between the ground state and first vibrational state is $0.35\,eV$. Take the mass of the hydrogen atom to be approximately the mass of the proton $m_H \approx 938\,\mathrm{MeV}/c^2$

- Determine the effective spring constant k express your answer in $eV/(nm^2)$ (Hint, what is the energy between the ground state and first excited states of the harmonic oscillator.)
- Write down the lowest order wave function of the hydrogen atom and make a qualitative sketch of the wave functions for the ground state. (You do not have to substitue numbers in this part)
- Determine the variance in position (i.e. Δx of the hydrogen atom). Determine your answer first analytically and then substitute numbers.
- Determine the average potential energy felt by the hydrogen atom in eV. (Work analytically then substitute numbers)
- Make an estimate for the average kinetic energy of the hydrogen atom in the well and estimate the velocity of the H. Compare these estimates to the kinetic energy and velocities of an electron in a Bhor orbit (Hint. $\beta = \alpha$, Hmmm, maybe I better add the Bohr model formulas to the formula sheet.)

A. The spacing between the ground and first excited state is $\Delta E = 0.35\,\text{eV}$. According to the energy levels of the spring, difference between $n = 1$ ($E = 3/2\,\hbar\omega_o$) and $n = 0$ ($E = 1/2\,\hbar\omega_o$)

$$\hbar\omega_o = \Delta E$$

So

$$\sqrt{\frac{k}{M_H}} = (\Delta E)/\hbar$$

and

$$\begin{aligned} k &= Mc^2 \frac{(\Delta E)^2}{(\hbar c)^2} \\ &= 938\ \times 10^6\,\text{eV}\,(0.35 eV)^2/(197\,\text{eV}\,nm)^2 \\ &= 2960\,\frac{eV}{nm^2} \end{aligned}$$

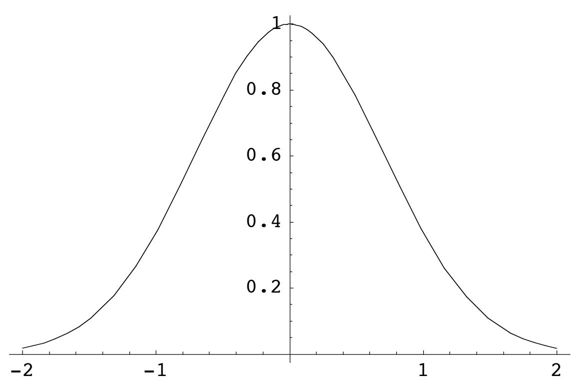
B. The potential is a spring, the functiion wave function is

$$\psi = \frac{1}{\pi^{1/4}}\frac{1}{\sqrt{L}}e^{-y^2/2} \tag{4}$$

with

$$L = \sqrt{\frac{\hbar}{M\omega_o}} = \sqrt{\frac{(\hbar c)^2}{Mc^2\hbar\omega_o}} = \sqrt{\frac{(197\text{eV nm})^2}{(938\times 10^6\,eV)(0.35\,eV)}} = 0.01\,\text{nm} \tag{5}$$

or about 20% of a Bohr radius.



C. To compute the variance in the position we note that $\langle x \rangle = 0$ for a spring and

$$\begin{aligned} \langle x^2 \rangle &= \int_{-\infty}^{\infty} dx\, x^2\, |\psi|^2 \\ &= L^2 \int dx \left(\frac{x}{L}\right)^2 \frac{1}{L\sqrt{\pi}} e^{-y^2} \\ &= L^2 \int du\, u^2 \frac{1}{\sqrt{\pi}} e^{-u^2} \\ &= \frac{L^2}{2} \qquad \text{The last integral would be given} \end{aligned}$$

So

$$\Delta x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \frac{L}{\sqrt{2}} = 0.007\,\text{nm}$$

C.

The average kinetic energy is

$$\begin{aligned} \langle PE \rangle &= \frac{1}{2} k \left\langle x^2 \right\rangle \\ &= \frac{1}{2} k L^2/2 \\ &= \frac{\hbar\omega_o}{4} \\ &= 0.35\,\text{eV}/4 \end{aligned}$$

The second to last line follows from the definitions of ω_o and L given above.

D.

$$\begin{aligned} \langle KE \rangle &\sim \langle PE \rangle \\ &\sim 0.35\,\text{eV}/4 \end{aligned}$$

For the velocity

$$\frac{1}{2} M v^2 \sim \langle KE \rangle$$

so

$$\begin{aligned} v &\sim \sqrt{\frac{2\,\langle KE \rangle}{M}} \\ v/c &\sim \sqrt{\frac{2\,\langle KE \rangle}{Mc^2}} \\ v/c &\sim \sqrt{\frac{2(0.35\,\text{eV})/4}{938 \times 10^6 \text{eV}}} \\ &\sim 1.4 \times 10^{-5} \end{aligned}$$

For the electron in hydrogen

$$KE \sim 13.6\,\text{eV} \qquad v/c \sim 1/(137) \tag{6}$$

So the velocity of the Hydrogen atom is moving approximately 500 times slower than the electron going around the hydrogen atom

- Write down the all the levels which are degenerate with $3s$ state of the hydrogen atom. (You do not have to bother yourself with spin in this problem. Just let me know if you are doing this problem with or without spin)
- Write down the $2p$ hydrogen wave function and verify that it is correctly normalized. Take the magnetic quantum number $m = 1$
- Make a qualitative sketch for the radial $2p$ wave function and also the associated probability.
- Determine the most likely position to find the electron in the $2p$ state.
- Make an estimate for the kinetic energies and compare your results to the Bohr model for $n = 2$. (Again, maybe I better add the Bohr model formulas) ;

A. These states neglecting spin are are: 3s states, $(n = 3, l = 0, m = 0)$, 3p states $(n = 3, l = 1, m = -1, 01)$, 3d states, $(n = 3, l = 2, m = -2, -1, 0, 1, 2)$.

B. For the 2p wave function we hve

$$\psi(r,\theta,\phi) = R_{nl}(r)Y_{lm}(\theta,\phi) \tag{7}$$

with $n = 2$ and $l = 1$ and $m = -1, 0, 1$. So then we have

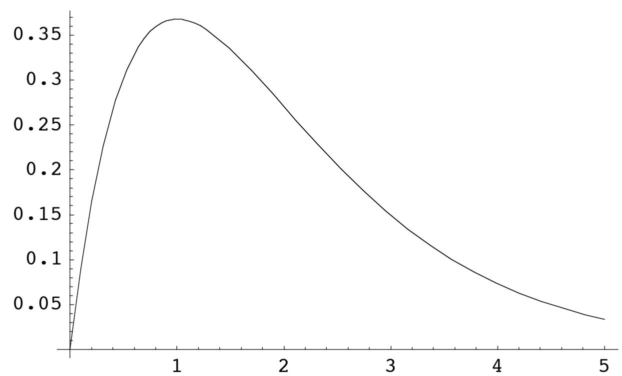
$$R_{21}(r) = \frac{1}{2\sqrt{6}}\frac{1}{a_0^{(3/2)}}\frac{r}{a_0}e^{-r/(2a_0)} \tag{8}$$

To verify that it is correctly normalized we need to show that

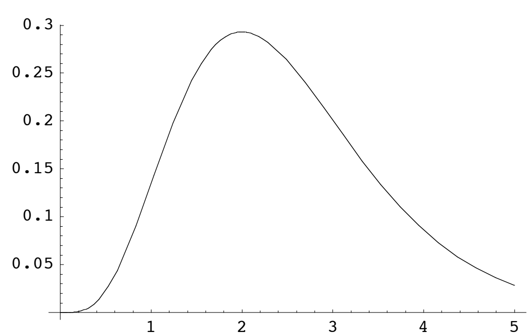
$$\begin{aligned}
\int_0^\infty dr(rR)^2 &= \int r^2 dr\, \frac{1}{24}\frac{1}{a_0^3}\left(\frac{r}{a_0}\right)^2 e^{-\frac{r}{a_0}} \\
&= \frac{1}{24}\int du\, u^4 e^{-u} \qquad \text{with } u = r/a_0 \\
&= 1/(24) \times 4! \qquad (\text{we use } I_n = \textstyle\int duu^n e^{-\alpha u} = \frac{n!}{\alpha^{(n+1)}}\ ) \\
&= 1 \qquad \text{The integral would be given}
\end{aligned}$$

B.

The wave function $R(r)$is schematically like this



The probability is $P(r) = |rR(r)|^2$ is schematically plotted like this



C.

To find the most likely position we write

$$P(r) = |rR(r)|^2 = \text{Const}\, r^4 e^{-r/a_0} \tag{9}$$

Then the most likely value is determined by maximizing $P(r)$.

$$\frac{dP}{dr} = C\left(4r^3 e^{-r/a_0} + r^4 e^{-r/a_0}(-\frac{1}{a_0})\right) = 0 \tag{10}$$

So solving for r we have

$$Cr^3 e^{-r/a_0}\left(4 - \frac{r}{a_0}\right) \quad = \quad 0$$

and

$$r = 4a_0 \tag{11}$$

Which is exactly the Bohr radius for $n = 2$.

$$r_n^2 = n^2 a_0 \tag{12}$$

D.

To estimate the momentum

$$p \sim \frac{\hbar}{r} \tag{13}$$

So KE

$$\begin{aligned} KE &\sim \frac{p^2}{2M} \\ &\sim \frac{\hbar^2}{2Mr^2} \\ &\sim \frac{\hbar^2}{2Ma_0^2}\frac{1}{16} \\ &\sim 13.6\,\text{eV}/16 \end{aligned}$$

The Bohr model KE is

$$KE = 13.6\,\text{eV}/2^2 \tag{14}$$

Though I would just be happy to see that you say that they are similar. in magnitude.