Problems:

10.8, 10.10, 10.14, 10.15, 10.16, 10.18, 10.21

- Derive Eq. (20) from Eq. (18) in this formula sheet. Determine the analgous equation for $1D$
- We place N electrons in a one dimensional box of length L
  - Show that the total energy

$$E_{\text{tot}} = 2\,L\,\frac{\hbar^2}{6\,m\pi}\,k_F^3 \tag{1}$$

    The leading factor of 2 comes from the sum over spins
  - Show that the energy per particle (or the average energy)

$$\frac{E_{\text{tot}}}{N} = \frac{1}{3}\,\frac{\hbar^2 k_F^2}{2\,m} \tag{2}$$

Statistical Mechanics

- For a system interacting in with a heat bath, the probability that the system will have energy E is

$$P(E) \propto e^{-\frac{E}{k_B T}} \tag{3}$$

  We will usually use this for a single particle interacting with the bath.

- For an ideal gas, the particles velocities are distributed according to the Maxwell-Boltzman velocity distributions

$$P(v)\underbrace{d^3\mathbf{v}}_{dv_x dv_y dv_z} = \left(\frac{M}{2\pi k_B T}\right)^{3/2} e^{-\frac{Mv^2}{2k_BT}}\,d^3\mathbf{v} \tag{4}$$

  This is the probability to find a particle with velocity $\mathbf{v}$ in a cell of size dv_x, dv_y, dv_z. If you do not care about the direction then we can integrate over the sphere in velocity space to find

$$P(v)\,dv = \left(\frac{M}{2\pi k_B T}\right)^{3/2} e^{-\frac{Mv^2}{2k_BT}}\,4\pi v^2\,dv \tag{5}$$

  This is the probability that an ideal gas will have speed between v and $v + dv$

- The number of particles per unit volume with speed between v and $v+dv$ is found by multiplying this probability with the total number of particles per volume N/V

$$n(v)\,dv = \frac{N}{V}\left(\frac{M}{2\pi k_B T}\right)^{3/2} e^{-\frac{\frac{1}{2}Mv^2}{k_BT}}\,4\pi v^2\,dv \tag{6}$$

Classical Gasses vs. Quantum Gasses

- For a gas at finite temperature the de Broglie wavelength is of order

$$\lambda_{\text{thermal}} \sim \frac{\hbar}{\sqrt{m\,k_B T}} \tag{7}$$

- The spacing between particles is of order

$$d \sim \left(\frac{V}{N}\right)^{1/3} \tag{8}$$

  where V is the volume of the box and N is the total number of particles in the box.

- The condition that we can treat the molecules of the gas as classical is

$$\lambda_{\text{thermal}} \ll d \tag{9}$$

Mode Counting

- $g(\mathbf{k})$ (in $3D$ for example) is the number of modes with momentum vector between (k_x, k_y, k_z) and $(k_x + dk_x, k_y + dk_y, k_z + dk_z)$. [1]. Similarly, in 2D $g(\mathbf{k})$ is the number of modes with momentum between (k_x, k_y) and $(k_x + dk_x, k_y + dk_y)$. In class we used the spacing between modes $\Delta k = \frac{\pi}{L}$ for a particle in the box. We have,

$$g(k)\,dk \;=\; 2\frac{dk}{\Delta k} = 2\,\frac{L\,dk}{\pi} \qquad \text{1D} \tag{10}$$

$$g(\mathbf{k})\,d^2\mathbf{k} \;=\; 2\frac{d^2k}{(\Delta k)^2} = 2\,\frac{A\,d^2k}{\pi^2} \qquad 2D \tag{11}$$

$$g(\mathbf{k})\,d^3\mathbf{k} \;=\; 2\frac{d^3k}{(\Delta k)^3} = 2\,\frac{V\,d^3k}{\pi^3} \qquad 3D \tag{12}$$

  The leading factor of two is the sum over spins.

- If we are only interested in the magnitude of k and not the directions we should integrate over $\frac{1}{4}$ of the circle (in 2D) and the $\frac{1}{8}$ sphere in $3D$

$$g(k)\,dk \;=\; 2\,\frac{L}{\pi}\,dk \qquad \text{1D} \tag{13}$$

$$g(k)\,dk \;=\; 2\,\frac{A}{\pi^2}\,\frac{2\pi k}{4}dk \qquad 2D \tag{14}$$

$$g(k)\,dk \;=\; 2\,\frac{V}{\pi^3}\,\frac{4\pi k^2}{8}dk \qquad 3D \tag{15}$$

  i.e. $g(k)$ is the number of modes with momentum magnitude between k and $k + dk$

Fermi Gas

- Electrons in a fermi gas fill all modes one by one until all particles are used up. The last state filled is known as the Fermi momentum. You should feel comfortable deriving the following results, especially in $1D$.
  - In one dimension this becomes the statement

$$N = 2\,L\frac{k_F}{\pi} \tag{16}$$

    The factor of two takes into account the spin of the electron. N is the number of electrons and L is the length of the box
  - In two dimensions this is the statement

$$N = 2\,A\frac{\pi k_F^2}{4\pi^2} \tag{17}$$

    The two takes into account the spin. N is the number of electrons and A is the area of the box
  - In three dimensions this the statement

$$N = 2\,V\frac{\frac{4}{3}\pi k_F^3}{8\pi^3} \tag{18}$$

    The two takes into account the spin. N is the number of electrons and V is the volume of the box

  The Fermi energy (in all dimensions) is

$$E_F = \frac{\hbar^2 k_F^2}{2M} \tag{19}$$

  simply the energy of the last orbital

- For three dimesions using Eq. (18) this can be written

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{20}$$

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[1] Be aware that we will take k_x, k_y, k_z as positive. In later courses people may take k_x, k_y, k_z positive and negative which makes some of these formulas change.