Problems:
3.4

- What is the most probable frequency for a black body photon at the following temperatures: (a) 3K microwave background (b) tungsten filament at 3000 K. (c) A fusion reaction at 10^7 K. Give your answer in eV.
- For what range of temperatures is the most probable frequency within the visible range $\lambda = 400\,\text{nm} - 700\,\text{nm}$.
- Estimate the energy density of a nucleus. Take Au which has 197 (protons + neutrons) and a radius of $R_{Au} = 5.5\,\text{fm}$. ($1\,\text{fm} = 10^{-15}\,\text{m}$. This unit of length is known as a femptometer or fermi.) Each proton or neutron has rest energy $E = mc^2 \approx 938\,\text{MeV}$. Give your answer in units of MeV/fm^3. At what temperature (or k_BT) would the energy per unit volume of a black body equal the energy density of a nucleus. Give your answer in MeV.

Mode Counting

- $g(\mathbf{k})$ (in $3D$ for example) is the number of modes with momentum vector between (k_x, k_y, k_z) and $(k_x+dk_x, k_y+dk_y, k_z+dk_z)$. [1]. Similarly, in 2D $g(\mathbf{k})$ is the number of modes with momentum between (k_x, k_y) and (k_x+dk_x, k_y+dk_y). In class we used the spacing between modes $\Delta k = \frac{\pi}{L}$ for a particle in the box. We have,

$$g(k)\,dk = 2\frac{dk}{\Delta k} = 2\,\frac{L\,dk}{\pi} \qquad \text{1D} \tag{1}$$

$$g(\mathbf{k})\,d^2\mathbf{k} = 2\frac{d^2k}{(\Delta k)^2} = 2\,\frac{A\,d^2k}{\pi^2} \qquad 2D \tag{2}$$

$$g(\mathbf{k})\,d^3\mathbf{k} = 2\frac{d^3k}{(\Delta k)^3} = 2\,\frac{V\,d^3k}{\pi^3} \qquad 3D \tag{3}$$

The leading factor of two is the sum over spins.

- If we are only interested in the magnitude of k and not the directions we should integrate over $\frac{1}{4}$ of the circle (in 2D) and the $\frac{1}{8}$ sphere in $3D$

$$g(k)\,dk = 2\,\frac{L}{\pi}\,dk \qquad \text{1D} \tag{4}$$

$$g(k)\,dk = 2\,\frac{A}{\pi^2}\,\frac{2\pi k}{4}dk \qquad 2D \tag{5}$$

$$= A\frac{k\,dk}{\pi} \tag{6}$$

$$g(k)\,dk = 2\,\frac{V}{\pi^3}\,\frac{4\pi k^2}{8}dk \qquad 3D \tag{7}$$

$$= V\frac{k^2\,dk}{\pi^2} \tag{8}$$

i.e. $g(k)$ is the number of modes with momentum magnitude between k and $k+dk$

Fermi Gas

- Electrons in a fermi gas fill all modes one by one until all particles are used up. The last state filled is known as the Fermi momentum. You should feel comfortable deriving the following results, especially in $1D$.
  - In one dimension this becomes the statement

$$N = 2\,L\frac{k_F}{\pi} \tag{9}$$

    The factor of two takes into account the spin of the electron. N is the number of electrons and L is the length of the box
  - In two dimensions this is the statement

$$N = 2\,A\frac{\pi k_F^2}{4\pi^2} \tag{10}$$

    The two takes into account the spin. N is the number of electrons and A is the area of the box

– In three dimensions this the statement

$$N = 2\,V\,\frac{\frac{4}{3}\pi k_F^3}{8\pi^3} \tag{11}$$

The two takes into account the spin. N is the number of electrons and V is the volume of the box The Fermi energy (in all dimensions) is

$$E_F = \frac{\hbar^2 k_F^2}{2M} \tag{12}$$

simply the energy of the last orbital

– For three dimesions using the above formulas this can be written

$$E_F = \frac{\hbar^2}{2m}\left(\frac{3\pi^2 N}{V}\right)^{2/3} \tag{13}$$

– You should be able to understand the origin of this qualitatively. The typical wavelength is

$$\lambda \sim \left(\frac{V}{N}\right)^{1/3} \tag{14}$$

The typical energy is then

$$\frac{p^2}{2m} \sim \frac{\hbar^2}{2m\lambda^2} \sim \frac{\hbar^2}{2M}\left(\frac{N}{V}\right)^{2/3} \tag{15}$$

Black Body Radiation

- For a blackbody emitting radiation, The number of photons in the box with frequency between ω and $\omega + d\omega$ is

$$N(\omega)\,d\omega = \frac{V}{\pi^2 c^3}\,\frac{\omega^2 d\omega}{e^{\frac{\hbar\omega}{k_B T}} - 1} \tag{16}$$

The most probable frequency for a photon is found by maximizing this function

$$\hbar\omega_m = 1.6\,k_B T \tag{17}$$

- The typical wavelength is of order

$$\lambda \sim \frac{\hbar c}{k_B T} \tag{18}$$

- The number of photons per unit volume

$$\frac{N}{V} = 0.244\left(\frac{\hbar c}{k_B T}\right)^3 \tag{19}$$

You should be able to give an estimate for this result from the following line of reasoning

$$N \sim \frac{V}{\lambda^3} \sim V\left(\frac{k_B T}{\hbar c}\right)^3 \tag{20}$$

- The energy density

$$u_E \equiv \frac{E}{V} = (k_B T)\left(\frac{k_B T}{\hbar c}\right)^3 \frac{\pi^2}{15} \tag{21}$$

You should be able to estimate this as

$$E \sim k_B T\,N \sim (k_B T)\left(\frac{k_B T}{\hbar c}\right)^3 \tag{22}$$

- The energy radiated per unit area per unit time is known as the emissivity

$$e = \text{Energy emitted per unit area per time} = u_E \, \frac{c}{4} \tag{23}$$

For a blackbody

$$e = \sigma T^4 \qquad \sigma = \frac{\pi^2}{60} \frac{c k_B^4}{(\hbar c)^3} = 5.67 \times 10^{-8} \, \frac{W}{m^2 K^4} \tag{24}$$

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[1] Be aware that we will take k_x, k_y, k_z as positive. In later courses people may take k_x, k_y, k_z positive and negative which makes some of these formulas change.