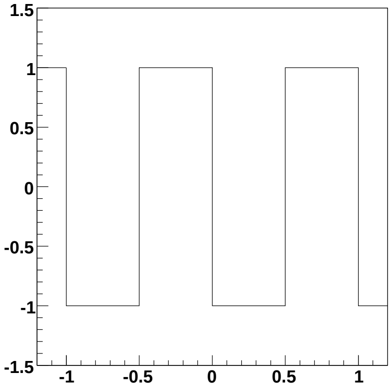

Figure 1: Take the size of the box to be $-0.5$ to $0.5$.

**Problem 1.** Consider the function shown in figure one. Determine its Fourier series expansion.

To determine the fourier expansion

$$\begin{aligned} c_0 &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)dx & (1)\\ &= 0 & (2)\end{aligned}$$

This is because the function is odd. I have attatched a short note on even odd, functions if you do not understand this. Then

$$\begin{aligned} \frac{A_n}{2} &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\cos(k_n x) & (3)\\ &= 0 & (4)\end{aligned}$$

This is because $\cos(k_n x)$ is an even function, while the function $f(x)$ is odd, the resulting integrand $f(x)\cos(k_n x)$ is odd. Finally $B_n$

$$\frac{B_n}{2} = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\sin(k_n x) \quad (5)$$

The integrand is a product of an even function, since $f(x)$ is odd and $\sin(k_n x)$ is odd. We can change the integration from $0 \ldots L/2$ and multiply by 2.

$$\frac{B_n}{2} = \frac{2}{1}\int_{0}^{1/2} f(x)\sin(k_n x) \quad (6)$$

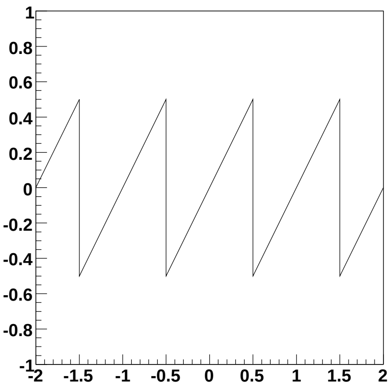

Figure 2: Take the size of the box to be $-0.5$ to $0.5$.

where we have inserted the size of the box $L = 1$. Then for $x$ between $0..1/2$ we have $f(x) = -1$

$$
\begin{aligned}
\frac{B_n}{2} &= \frac{2}{1}\int_0^{1/2} -\sin(k_n x) & (7)\\
&= 2\frac{\cos(k_n x)}{k_n}\bigg|_0^{1/2} & (8)\\
&= -\frac{2 - 2\cos(\pi n)}{2\pi n} & (9)
\end{aligned}
$$

where in the last line we substitute $k_n = (2\pi n)/L$ Then examining the graph of $\cos(x)$ we see that

$$\cos(\pi n) = (-1)^n \qquad \text{for} \qquad n = 1, 2, 3\ldots \tag{10}$$

So $B_n$ alternates

$$B_n = -\frac{4}{\pi}, 0, -\frac{4}{3\pi}, 0, -\frac{4}{5\pi}, \ldots \qquad \text{for} \qquad n = 1, 2, 3, \ldots \tag{11}$$

See the additional handout from section about how well this series expansion works

**Problem 2.** Consider the function shown in figure two. Determine its Fourier series expansion. A identical analysis to the last problem shows that

$$c_0 = 0 \tag{12}$$

and

$$A_n = 0 \tag{13}$$

Bear in mind that this is only for odd functions $f(x)$. For even functions this would not be the case – $B_n$ would vanish and not $A_n$. To compute $B_n$ for this odd $f(x)$ we follow similar steps as last time

$$\begin{aligned} \frac{B_n}{2} &= \frac{1}{L}\int_{-L/2}^{L/2} f(x)\sin(k_n x) \quad (14)\\ &= \frac{1}{1}\int_{-1/2}^{1/2} x\sin(k_n x) \quad (15) \end{aligned}$$

To do this integral we integrate by parts, setting $dv = \sin(k_n x)$ , $v = -\cos(k_n x)/k_n$ and $u = x$

$$\begin{aligned} \frac{B_n}{2} &= -\left.\frac{x\cos(k_n x)}{k_n}\right|_{-1/2}^{1/2} - \frac{1}{k_n}\int_{-1/2}^{1/2}(-\cos(k_n x))dx \quad (16)\\ &= 2\times\frac{1}{2}\frac{1}{k_n}\cos(k_n/2) + \frac{2}{k_n}\sin(k_n/2) \quad (17)\\ & \quad (18) \end{aligned}$$

The last line follows because $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$. The result further simplifies.

$$\frac{B_n}{2} = \frac{(-1)^{n+1}}{2\pi n} + 0 \tag{19}$$

The last line follows by realizing that $\sin(k_n/2) = \sin(\pi n) = 0$ and using the result from above

$$\cos(\pi n) = (-1)^n \qquad \text{for} \qquad n = 1, 2, 3\ldots \tag{20}$$

So

$$B_n = \frac{1}{\pi}, -\frac{2}{2\pi}, +\frac{1}{3\pi}, \ldots \qquad \text{for} \qquad n = 1, 2, 3, \ldots \tag{21}$$

To see how well we are doing look at the supplemental page.

# 1 Even-odd functions

- Even functions are the same with respect to flipping $x \to -x$ (like $\cos(x)$
$$f(-x) = f(x) \tag{22}$$
- Odd functions are the same up to a sign $x \to -x$ (like $\sin(x)$)
$$g(-x) = -g(x) \tag{23}$$
- Integrals around zero of odd functions are zero
$$\int_{-L}^{L} g(x) = 0 \tag{24}$$
for example $\int_{-L}^{L} \sin(x) = -\cos(x)|_{-L}^{L} = 0$
- An even times an odd function is odd.
$$\begin{aligned} \text{even} \times \text{even} &= \text{even} && (25) \\ \text{even} \times \text{odd} &= \text{odd} && (26) \\ \text{odd} \times \text{odd} &= \text{even} && (27) \\ & && (28) \end{aligned}$$
- Thus
$$\int_{-L/2}^{L/2} dx\, x \cos(\pi x) = 0 \tag{29}$$
since $x$ is odd and $\cos(\pi x)$ is even. See the additional handout from section about how well this series expansion works.