0.1 Fourier Series

Any periodic function with period L can be written as a sum of sins and coss. (A periodic function which repeats itself over a distance L. We will start the function at $-L/2$ and go up to $L/2$ where upon the function repeats.)

$$f(x) = c_0 + \sum_{n>0} A_n \cos(k_n x) + B_n \sin(k_n x) \tag{1}$$

with

$$k_n = \frac{2\pi n}{L}$$

The coefficients are defined by averages of the function over the box

$$\langle \ldots \rangle = \frac{1}{L} \int_{-L/2}^{L/2} \ldots \tag{2}$$

where $\ldots$ means any function. It should be clear that

$$\langle \cos(k_n x) \rangle = 0 \qquad \langle \sin(k_n x) \rangle = 0 \tag{3}$$

since the cos is sometimes up and sometimes down – its average value is zero. Averaging both sides of Eq.**??** we have

$$\begin{aligned}
\langle f(x) \rangle &= \left\langle c_0 + \sum_{n>0} A_n \cos(k_n x) + B_n \sin(k_n x) \right\rangle && (4)\\
&= c_0 \langle 1 \rangle + \sum_{n>0} A_n \langle \cos(k_n x) \rangle + B_n \langle \sin(k_n x) \rangle && (5)\\
\langle f(x) \rangle &= c_0 && (6)
\end{aligned}$$

i.e. that

$$c_0 = \frac{1}{L} \int_{-L/2}^{L/2} f(x) \tag{7}$$

The remaining coefficients follow from a similar procedure. The results follow from the following integrals

$$\begin{aligned}
\langle \cos(k_n x) \cos(k_m x) \rangle &= \begin{cases} 0 & \text{for} \quad k_n \neq k_m \\ \frac{1}{2} & \text{for} \quad k_n = k_m \end{cases} && (8)\\
\langle \sin(k_n x) \sin(k_m x) \rangle &= \begin{cases} 0 & \text{for} \quad k_n \neq k_m \\ \frac{1}{2} & \text{for} \quad k_n = k_m \end{cases} && (9)\\
\langle \cos(k_n x) \sin(k_m x) \rangle &= 0 && (10)
\end{aligned}$$

which you will prove below. For instance to determine A_l one multiplies by $\cos(k_l x)$ and averages both sides

$$\begin{aligned}\langle f(x)\cos(k_l x)\rangle &= \left\langle c_0\cos(k_l x) + \sum_n A_n\cos(k_n x)\cos(k_l x) + B_n\sin(k_n x)\cos(k_l x)\right\rangle && (11)\\ &= A_l\,\langle\cos(k_l x)\cos(k_l x)\rangle && (12)\\ &= \frac{A_l}{2} && (13)\end{aligned}$$

Or more explicitly

$$\frac{A_l}{2} = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\cos(k_l x) \qquad (14)$$

Similarly to determine B_l one would multiply by $\sin(k_l x)$ and average.

To summarize we have the following coefficients:

$$\begin{aligned} c_0 &= \langle f\rangle = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\,dx && (15)\\ \frac{A_n}{2} &= \langle f\cos(k_n x)\rangle = \frac{1}{L}\int f(x)\cos(k_n x) && (16)\\ \frac{B_n}{2} &= \langle f\sin(k_n x)\rangle = \frac{1}{L}\int_{-L/2}^{L/2} f(x)\cos(kx) && (17)\end{aligned}$$

0.2 Complex Fourier series

Again any periodic function can be written as a sum of sins and coss. Rather than using sins and coss we can use

$$e^{\pm ik_n x} = cos(k_n x) \pm isin(k_n x) \qquad (18)$$

and the inverse relations

$$\begin{aligned}\cos(k_n x) &= (e^{+ik_n x} + e^{-ik_n x})/2 && (19)\\ \sin(k_n x) &= (e^{+ik_n x} - e^{-ik_n x})/2i && (20)\end{aligned}$$

The result is written like this

$$f(x) = \frac{1}{L}\sum_n \hat{f}(k_n)e^{ik_n x} \qquad (21)$$

Physically we have

$$\hat{f}(k_n) = \text{How much of the n-th wavelength is in f(x)} \tag{22}$$

Note that in writing this we have a sum over all n both positive and negative

$$k_n = \frac{2\pi n}{L} \qquad \ldots - 2, -1, 0, 1, 2, \ldots \tag{23}$$

You will show in the homework that we have the following integral

$$\int_{-L/2}^{L/2} e^{ik_n x} = \begin{cases} 0 & \text{for} \quad k_n \neq 0 \\ L & \text{for} \quad k_n = 0 \end{cases}$$

Then we have the following relation which you will also prove in the homework

$$\hat{f}(k_n) = \int_{-L/2}^{L/2} f(x) e^{-ik_n x} \tag{24}$$

Also note that $k_0 = 0$ and for $k_0 = 0$ the $\hat{f}$ is just the integral of $f(x)$

$$\hat{f}(0) = \int_{-L/2}^{L/2} f(x) \tag{25}$$

0.3 Fourier Transforms

So far we have delt with periodic waves. In general a wave can be a pulse and then is not periodic. To generalize our discussion to arbitrary functions we take the size of the box very large $L \to \infty$. As we do this the spacing between the k_n gets closer and closer. Since $k_n = 2\pi n/L$

$$\begin{aligned} \delta k &= k_{n+1} - k_n && (26) \\ &= \frac{2\pi}{L}(n+1) - \frac{2\pi}{L} n && (27) \\ &= \frac{2\pi}{L} && (28) \end{aligned}$$

Then k_n becomes a coninuous variable. Since $k_n = 2\pi n/L$

$$dk = \frac{2\pi}{L} dn \tag{29}$$

The sum over n becomes an integral

$$\sum_n = \int dn = \int \frac{Ldk}{2\pi} \tag{30}$$

This is important

$$\sum_n \to \int \frac{Ldk}{2\pi} \tag{31}$$

Then we have

$$\begin{aligned}
f(x) &= \frac{1}{L} \sum_n \hat{f}(k_n) e^{+ik_n x} && (32)\\
&= \frac{1}{L} \int \frac{Ldk}{2\pi} \hat{f}(k) e^{+ikx} && (33)\\
&= \int_{-\infty}^{\infty} \frac{dk}{2\pi}\, \hat{f}(k)\, e^{+ikx} && (34)
\end{aligned}$$

Putting the formulas together

$$\begin{aligned}
\hat{f}(k) &= \int_{-\infty}^{\infty} f(x) e^{-ikx} && (35)\\
f(x) &= \int \frac{dk}{2\pi} e^{+ikx}\, \hat{f}(k) && (36)\\
& && (37)
\end{aligned}$$

Physically $\hat{f}(k)$ records how much of a wave with wavelength

$$k = \frac{2\pi}{\lambda} \tag{38}$$

is inside the function $f(x)$.

- This problem works through some integrals and gives practice using $e^{\pm ikx} = \cos(kx) \pm i\sin(kx)$. In this problem $k_n = 2\pi n/L$ and $k_m = 2\pi m/L$ with n and m integers $\ldots - 2, -1, 0, 1, 2 \ldots$
  - Show that

    $$\int_{-L/2}^{L/2} e^{ik_n x} = \begin{cases} 0 & \text{for} \qquad k_n \neq 0 \\ L & \text{for} \qquad k_n = 0 \end{cases}$$

    You can do this in two ways by writing the exponential as $\cos \pm i \sin$ or just integrating the exponential.
  - Starting from the discrete form of the fourier transform, i.e.

    $$f(x) = \frac{1}{L}\sum_n \hat{f}(k_n) e^{+ik_n x} \tag{39}$$

    use the integral above to show that

    $$\hat{f}(k_n) = \int_{-L/2}^{L/2} f(x) e^{-ik_n x}$$

    (Hint: Multiply both sides of Eq. **??** by $e^{-ik_n x}$ and integrate)
  - Show by direct integration that

    $$\frac{1}{L}\int_{-L/2}^{L/2} \cos(k_n x)\cos(k_m x) = \begin{cases} 0 & \text{for} \qquad k_n \neq k_m \\ \frac{1}{2} & \text{for} \qquad k_n = k_m \end{cases}$$

    Do this by using the relations and the integral above

    $$\cos(kx) = (e^{ikx} + e^{-ikx})/2 \qquad \sin(kx) = (e^{+ikx} - e^{-ikx})/2i$$

  - Now show the following integrals by direct integration

    $$\frac{1}{L}\int_{-L/2}^{L/2} \cos(k_n x)\sin(k_m x) = 0$$

    $$\frac{1}{L}\int_{-L/2}^{L/2} \sin(k_n x)\sin(k_m x) = \begin{cases} 0 & \text{for} \qquad k_n \neq k_m \\ \frac{1}{2} & \text{for} \qquad k_n = k_m \end{cases}$$

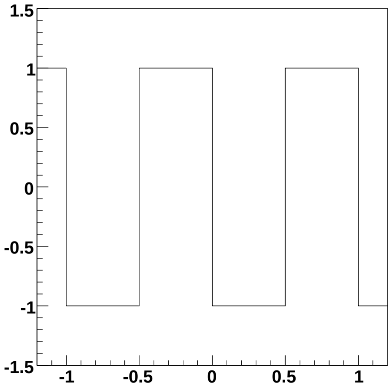

Figure 1: Take the size of the box to be -0.5 to 0.5.

- Consider the function shown in figure one. Determine its Fourier series expansion. First in terms of sin and cos and then in terms of $e^{ik_n x}$. That is determine the
$$c_0, A_n, B_n, \hat{f}(k_n)$$
If you can, find a computer running a plotting program (such as mathematica or maple) and plot the first couple of terms in this series using the cos and sin form.

- Consider the function shown in figure two. Determine its Fourier series expansion. First in terms of sin and cos and then in terms of $e^{ik_n x}$. That is determine the
$$c_0, A_n, B_n, \hat{f}(k_n)$$
If you can, find a computer running a plotting program (such as mathematica or maple) and plot the first couple of terms in this series. Hint $\int x\sin(x)$ can be integrated by parts by setting $u = x$ and $dv = \sin(x)$. Similarly for $\int x\cos(x)$ and $\int xe^{ikx}$

- Determine the Fourier transform $\hat{f}(k)$ of the following function.
$$f(x) = \frac{1}{2a} e^{-\frac{|x|}{a}} \tag{40}$$
Make a graph of $f(x)$ and $\hat{f}(k)$ for large and small a

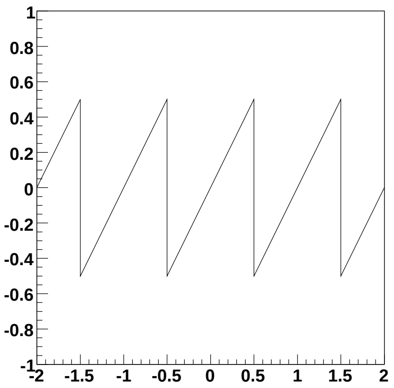

Figure 2: Take the size of the box to be -0.5 to 0.5.

- Determine the Fourier transform $\hat{f}(k)$ of the following function

$$f(x) = \frac{1}{\sqrt{2\pi a^2}} e^{-\frac{x^2}{2a^2}} \tag{41}$$

  You can use the gaussian integrals discussed in section

  - Start by showing that this function is normalized to one

$$\int_{-\infty}^{\infty} f(x) = 1$$

  - Next compute the fourier transform. Make a graph of $f(x)$ and $\hat{f}(k)$ for a large and for a small.

Gaussian Integrals

1. Gaussian integrals are handled as follows. Consider the following integral

$$I = \int_{-\infty}^{\infty} dx\, e^{-x^2} \tag{42}$$

Then

$$I^2 = \int dx\, e^{-x^2} \int dy\, e^{-y^2} = \int \int dx dy e^{-x^2+y^2} \tag{43}$$

Then we can change to polar coordinates to write this as

$$\begin{aligned} I^2 &= \int_0^{2\pi} d\theta \int_0^{\infty} r dr e^{-r^2} && (44) \\ &= (2\pi)\,\frac{1}{2} = \pi && (45) \end{aligned}$$

Thus we have

$$I = \sqrt{\pi} \tag{46}$$

2. Next consider a slight generaliztion of this

$$I(j) = \int_{-\infty}^{\infty} dx\, e^{-x^2+jx} \tag{47}$$

We may complete the square, $-x^2 + Jx = -(x - j/2)^2 + (j/2)^2$, and then shift the integration variables $y \equiv (x - j/2)$

$$I(j) = \int_{-\infty}^{\infty} e^{-x^2+jx} = \int dy e^{-y^2+(j/2)^2} = e^{(j/2)^2}\,\sqrt{\pi} \tag{48}$$

This formula works for j complex.

3. This allows us to compute integrals of polynomial time gaussian. We will need this later. For example

$$\begin{aligned} \int_{-\infty}^{\infty} dx\, x^2\, e^{-x^2} &= \left[\frac{\partial}{\partial j}\,\frac{\partial}{\partial j} \int dx\, e^{-x^2+jx}\right]_{j=0} = \left[\frac{\partial}{\partial j}\,\frac{\partial}{\partial j} e^{(j/2)^2}\sqrt{\pi}\right]_{j=0} && (49) \\ &= \frac{1}{2}\sqrt{\pi} && (50) \end{aligned}$$