1 Dot Product and Cross Products

- For two vectors, the dot product is a number

$$\mathbf{A}\cdot\mathbf{B} = AB\cos(\theta) = A_{\parallel}B = AB_{\parallel} \tag{1}$$

- For two vectors $\mathbf{A}$ and $\mathbf{B}$ the cross product $\mathbf{A}\times\mathbf{B}$ is a vector. The magnitude of the cross product

$$|\mathbf{A}\times\mathbf{B}| = AB\sin(\theta) = A_{\perp}B = AB_{\perp} \tag{2}$$

  The direction of the resulting vector is given by the right hand rule.

- A formula for the cross product of two vectors is

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \hat{\boldsymbol{\imath}} & \hat{\boldsymbol{\jmath}} & \hat{\boldsymbol{k}} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} \tag{3}$$

2 Work and Energy

- The work done by any force in going from position $\mathbf{r}_A$ up to position $\mathbf{r}_B$ is

$$W = \int_{\mathbf{r}_A}^{\mathbf{r}_B} \mathbf{F}\cdot d\mathbf{r} \tag{4}$$

  For a constant force the work is simply the dot product of the force with the displacement

$$W = \mathbf{F}\cdot\Delta\mathbf{r} = F\Delta r\,\cos(\theta) \tag{5}$$

  where θ is the angle between the displacement and the force vector.

- The work done by all forces

$$W_{\text{all}-\text{forces}} = \Delta K = K_f - K_i \tag{6}$$

  where the kinetic energy is

$$K = \frac{1}{2}mv^2 \tag{7}$$

- We classify forces as conservative (gravity springs) and non-conservative (friction). For conservative forces we can introduce the potential energy. The change in potential energy is minus the work done by the foce

$$\Delta U = U_2 - U_1 = -\int_1^2 \mathbf{F}\cdot d\mathbf{r} \tag{8}$$

- The force associated with a given potential energy is

$$F = -\frac{dU(x)}{dx} \tag{9}$$

- Then the fundamental work energy theorem can be written

$$W_{non-consv} + W_{\text{ext}} = \Delta K + \Delta U \tag{10}$$

  where ΔU is the change in potential energy of the system.

- If there are no external or dissipative forces then

$$E = K + U = \text{constant} \tag{11}$$

  You should understand the logic of how Eq. **??** leads to Eq. **??** and ultimately Eq. **??**.

- The potential energy depends on the force that we are considering:
  - For a constant gravitational force $F = mg$ we have
$$U = mgy \tag{12}$$
    where y is the vertical height measured from any agreed upon origin.
  - For a spring with spring constant k which is displaced from equilibrium by an amount x, we have a potential energy of
$$U = \frac{1}{2}kx^2 \tag{13}$$
  - For a particle a distance r from the earth the potential energy is
$$U = -\frac{GMm}{r} \tag{14}$$
- Power is defined as the rate at which work is done or the rate at which energy is transformed from one form to another.
$$P = \frac{dW}{dt} = \frac{dE}{dt} \tag{15}$$
  or
$$P = \mathbf{F} \cdot \mathbf{v} \tag{16}$$

3 Momentum

- The momentum of an object is
$$\mathbf{p} = m\mathbf{v} \tag{17}$$
  In terms of momentum Newtons Law can
$$\sum \mathbf{F} = \frac{d\mathbf{p}}{dt} \tag{18}$$
- The total momentum transferred to a particle by a force is the known as the impulse (or simply momentum transfer)
$$\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i = \int_{t_i}^{t_f} \mathbf{F} dt = \mathbf{J} \tag{19}$$
  If the force last a period Δt the average force is
$$\mathbf{F}_{\text{ave}} = \frac{\Delta\mathbf{p}}{\Delta t} \tag{20}$$
- For a system of particles with total mass M, we define the center of mass
$$\mathbf{x}_{cm} = \frac{\sum_i m_i \mathbf{x}_i}{M} \tag{21}$$
  For a continuous distribution of mass (e.g. a rod)
$$\mathbf{x}_{\text{cm}} = \frac{1}{M}\int \mathbf{x}\, dm \tag{22}$$
  See Example 9-16 and 9-17 for how to actually do these calculations.
- The total momentum is of a system of particles
$$\mathbf{P}_{\text{tot}} = \sum m_i \mathbf{v}_i = M\mathbf{v}_{\text{cm}} \tag{23}$$
  It should be clear how to derive this last equality by differentiating Eq. **??**

- Newtons laws for a system of particles is

$$\mathbf{F}_{\text{ext}} = \frac{d\mathbf{P}_{\text{tot}}}{dt} = M\mathbf{a}_{\text{cm}} \tag{24}$$

If the mass is changing e.g. in Rocket problems one should be careful drawing a picture of before and after a time Δt – see l19. You should feel comfortable deriving e.g. Eq. 9-19b of the book. See examples 9-19, 9-20.

- If there are no external forces in a system of particles then (from Eq. **??**) momentum is conserved

$$\mathbf{P}_{\text{tot}} = \text{Constant} \tag{25}$$

i.e. for $2 \to 2$ collisions

$$\mathbf{p}_A + \mathbf{p}_B = \mathbf{p}'_A + \mathbf{p}'_B \tag{26}$$

- If a collision is totally elastic (there is no internal disaptive or explosive forces). Energy is conserved

$$\frac{1}{2}m_A v_A^2 + \frac{1}{2}m_B v_B^2 = \frac{1}{2}m_A v_A'^2 + \frac{1}{2}m_B v_B'^2 \tag{27}$$

In *one dimensional elastic collisons* a simplified formula is equivalent to energy conservation

$$v_A - v_B = -(v'_A - v'_B) \tag{28}$$

- In an inelastic collision energy is not conserved.

4 Rotational Motion

4.1 Kinematics

- Use radians – most of these formulas assume it.
- The magnitude of the angular velocity and the angular acceleration of a rigid body are

$$\omega = \frac{d\theta}{dt} \qquad \text{and} \qquad \alpha = \frac{d\omega}{dt} \tag{29}$$

And these quantities do not depend on the radius (unlike velocity).

- The freqency and period (for ω constant is)

$$f = \frac{\omega}{2\pi} \qquad T = \frac{1}{f} \tag{30}$$

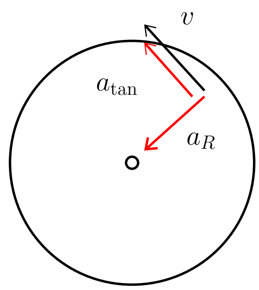
- The angular velocity $\boldsymbol{\omega}$ and angular acceleration $\boldsymbol{\alpha}$, point along the axis of rotation. The direction is given by the right hand rule.
- The velocity and tangential and radial accelerations are

$$v = R\omega \tag{31}$$

$$a_{\tan} = R\alpha \tag{32}$$

$$a_R = \frac{v^2}{R} = \omega^2 R \tag{33}$$

The total acceleration is a vector sum of thse For an object spinning counter clockwise and speeding up the picture is



- For *constant angular acceleration* the following formulas are valid (in analogy)

$$\theta = \theta_o + \omega_o t + \frac{1}{2}\alpha t^2 \tag{34}$$
$$\omega = \omega_o + \alpha t \tag{35}$$
$$\omega^2 = \omega_o^2 + 2\alpha\Delta\theta \tag{36}$$

4.2 1D-Dynamics and Energetics

- The torque is

$$\boldsymbol{\tau} = \mathbf{R} \times \mathbf{F} \tag{37}$$

When limitted to rotation in the xy plane we have

$$\boldsymbol{\tau} = \pm R_\perp F\,\hat{\boldsymbol{k}} = \pm R F_\perp\,\hat{\boldsymbol{k}} \tag{38}$$

$+\hat{\boldsymbol{k}}$ indicates a a counter-clockwise rotation while $-\hat{\boldsymbol{k}}$ indicates a clockwise rotation. For xy rotations the $\hat{\boldsymbol{k}}$ is usually not written down, but is understood.

- The moment of inertia of a solid body is

$$I = \sum_i m_i R_\perp^2 \qquad I = \int R_\perp^2\, dm \tag{39}$$

To compute the moment of inertia one can:

  - Perform the integral.
  - Break it up into pieces whose moment of inertial you know
  - Look it up (If I want you to look up I will provide a table)
  - Use the parallel axis theorem:

$$I_A = I_{cm} + Md^2 \tag{40}$$

  where d is the distance from the desired parallel axis to the center of mass.

- Torques create angular acceleration. For spinning around a natural axis of a body one has

$$\sum \boldsymbol{\tau} = I\boldsymbol{\alpha} \tag{41}$$

this applies around a fixed axis or around the center of mass if the body is accelerating.

- The rotational kinetic energy is

$$K_{\text{rot}} = \frac{1}{2} I\omega^2 \tag{42}$$

- If an object is moving there is rotational kinetic energy around the center of mass and there is translational kinetic energy

$$K_{\text{tot}} = \frac{1}{2} M v_{cm}^2 + \frac{1}{2} I_{\text{cm}}\omega^2 \tag{43}$$

- When a wheel is rolling is without slipping the point at the bottom of the wheel is instanteously not moving (it has non zero acceleration however). Thus what actually keeps a tire from slipping is the coefficient of static and not kinetic friction. If a an object is rolling *without slipping* we have

$$\omega = \frac{v_{\rm cm}}{R}$$

  Otherwise ω and $v_{\rm cm}$ are separate quantities to be determined by $F_{\rm net} = Ma_{\rm cm}$ and $\tau = I\frac{d\omega}{dt}$.

4.3 Angular Momentum

- The angular momentum of a rigid body rotating about a principle axis is

$$\mathbf{L} = I\boldsymbol{\omega} \tag{44}$$

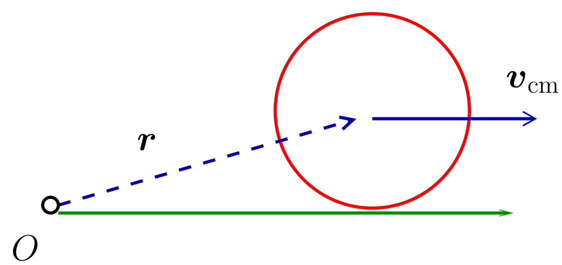
- The angular momentum of a particle is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} \qquad |\mathbf{L}| = r_\perp mv \tag{45}$$

- The total angular momentum of a system is the sum of the angular momenta of its different components. It depends on the axis of rotation For example:

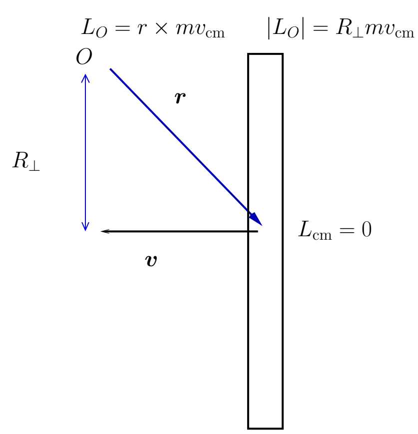
  1. Ball rolling – Calculate $L_{\rm cm}$ and $L_{\rm O}$

$$L_{\rm cm} = I_{\rm cm}\omega$$



$$L_O = \boldsymbol{r} \times M_{\rm tot}\boldsymbol{v}_{\rm cm} + I_{\rm cm}\omega$$

  2. Rod just moving to right with speed v. Calculate $L_{\rm cm}$ and $L_{\rm O}$



- The net external torque on a system (about a fixed axis or about the center of mass if the object is accelerating) determines the rate of change in angular momenta

$$\sum \tau_{\rm ext} = \frac{d\mathbf{L}_{\rm tot}}{dt} \tag{46}$$

- If there is no net external torque then the total angular momentum is conserved

$$\mathbf{L}_{\text{init}} = \mathbf{L}_{\text{final}} \tag{47}$$

5 Oscillations

- We derived several examples of small oscillations
  - For a mass connected to a spring the equation of motion become

    $$\frac{d^2x}{dt^2} = -\frac{k}{M}x \tag{48}$$

    You should know how to derive this using $F = Ma$.
  - Similarly we showed (using $\sin(\theta) \approx \theta$ for small angles) that for a small blob connected to a string of length l, The equation of motion is

    $$\frac{d^2\theta}{dt^2} = -\frac{g}{l}\theta \tag{49}$$

    You should know how to derive this from $F = ma$
  - Finally we showed (using $\sin(\theta) \approx \theta$ for small angles) that for a solid pendulum of moment of inertia I, pivoted a distance h above the center of mass the angle obeys

    $$\frac{d^2\theta}{dt^2} = -\frac{mgh}{I}\theta \tag{50}$$

    You should know how to derive this from $\tau = I\alpha$. Or memorize the period, etc if you must.
  - The generic formula is

    $$\frac{d^2u}{dt^2} = -\omega_o^2 u \tag{51}$$

    where u is the thing thats ocillating and ω_o is the angular oscillation frequency.
- The preceding equations are all the same with the subsitutions (e.g. $x \to \theta$ and $k/M \to g/\ell$). We will take the spring for simplicity but these remarks to apply to the other cases as well. The spring is released from position x_0 with velocity v_0 at time $t = 0$. The free constants in the general solution $x(t) = C_1 \cos(\omega_o t) + C_2 \sin(\omega_o t)$ are adjusted so that $x(0) = x_0$ and $\dot{x}(0) = v_0$. You should be able to show that in this case $C_1 = x_o$ and $C_2 = v_o/\omega_o$

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega_o}\sin(\omega_o t) \qquad \omega_o = \sqrt{\frac{k}{M}} \tag{52}$$

- It often instructive to write this in amplitude + phase form. We showed in class that Eq. **??** can be rewritten (you should be able to show this)

$$x(t) = A\cos(\omega t - \phi) \tag{53}$$

where

$$A = \sqrt{x_o^2 + (v_o/\omega_o)^2} \qquad \text{and} \qquad \phi = \tan^{-1}\left(\frac{v_o}{\omega_o x_o}\right) \tag{54}$$

- The frequency and period of the oscillation are

$$f = \frac{\omega_o}{2\pi} \qquad T = \frac{1}{f} \tag{55}$$

- Analgous formulas hold for the other cases. For example for a simple pendulum released from and initial angle θ_o with iniitial angular velocity $\Omega_o = \dot{\theta}(0)$ the analgous formulas are

$$\theta(t) = \theta_o \cos(\omega_o t) + \frac{\Omega_o}{\omega_o}\sin(\omega_o t) \qquad \omega_o = \sqrt{\frac{g}{l}} \tag{56}$$

- During simple harmonic motion, the energy of a spring changes between kinetic and potential energies. The total energy is constant
$$E = \frac{1}{2}Mv^2 + \frac{1}{2}kx^2 \tag{57}$$

- For a harmonic oscillator with non-zero damping force $F_D = -bv$ you should be able to derive the following equation of motion
$$\frac{d^2x}{dt^2} + \frac{b}{M}\frac{dx}{dt} + \frac{k}{M}x = 0 \tag{58}$$
which has a general solution
$$x(t) = Ae^{-\frac{b}{2m}\,t}\cos(\omega t - \phi) \qquad \text{where} \qquad \omega = \sqrt{\frac{k}{M} - \frac{b^2}{4m^2}} \tag{59}$$
The constants A and ϕ are as usual adjusted to reproduce the initial conditions. We will keep the discussion fairly elementary, at the level of Example 14-11 of the book.

- For a vertical spring, when mass is added, the equilbirium point is shifted downward (derive):
$$x_{\rm eq} = -\frac{mg}{k} \tag{60}$$
If we measure the deviation from this equilibrium point
$$y = x - x_{\rm eq} \tag{61}$$
we have the classic equation of motion (show)
$$\frac{d^2y}{dt^2} = -\frac{k}{M}y \tag{62}$$
The potential energy measures both the gravitational potential energy and the spring potential energy (show):
$$U = \frac{1}{2}ky^2 = \frac{1}{2}kx^2 + mgx + \text{constant} \tag{63}$$

6 Gravitation

- The universal law of gravitational attraction is a force attracting mass M with mass m.
$$F = \frac{GMm}{r^2} \tag{64}$$
The direction of this force is along the line joining the two particles and is always attractvie.

- You should be able to show that
$$g = \frac{GM_E}{R_E^2} \tag{65}$$

- You should be able to compute the properties of circular orbits in this kind of force field, e.g. The kinetic energy for an orbit of radius R.

- You should be able to compute the escape velocity from the earth etc.

7 Statics–Section 12-1, Section 12-2

- For static equilibrium one has only equation
$$\sum \mathbf{F}_{\boldsymbol{i}} = \mathbf{0} \qquad \sum \boldsymbol{\tau} = \mathbf{0} \tag{66}$$
This when carefully applied is all you need.