\[ \vec{A} \cdot \vec{B} = AB \cos \theta \]

Notation: \( \vec{A} \) denotes vector \( A \) w/ out arrow means \( |\vec{A}| \) \( \text{or} \) \( \sqrt{\vec{A} \cdot \vec{A}} = A \)

Written:

\[ \vec{A} \cdot \vec{B} = A_{||}B = A_xB_x + A_yB_y \]

**Work**

\[ W = \vec{F} \cdot \Delta x = F_{||} \Delta x = F \Delta x \cos \theta \]

**Analyzed Work**

\[ W_n = N \Delta x \cos \theta \]

\[ = 0 \quad \text{Since} \quad \theta = 90^\circ \]

\[ \omega_g = mg \Delta x \cos \theta < 0 \]

Since \( \theta > 90^\circ \)

\[ \omega_g = mg \Delta x \sin \theta \]
Now

\[ W_g = mg \Delta x \cos(90^\circ + \phi) \]

\[ = -mg \Delta x \sin \phi \]

\[ \frac{\Delta x}{\Delta y} \]

\[ W_g = -mg (y_2 - y_1) \]

A very general formula which we will see more of today.
What if \( F \) not constant or \( \theta \) not constant

\[
W = \sum_i F_i \cos \theta_i \, dl = \int_a^b F(l) \cos \theta(l) \, dl
\]

Alternatively

\[
W = \sum_i \vec{F} \cdot \vec{dl} = \int_a^b \vec{F} \cdot d\vec{l}
\]

This is known as a line integral

More math than needed:

\[
\vec{l} = x(l) \hat{i} + y(s) \hat{j}
\]

\[
\vec{dl} = \left( \frac{dx}{dl} \hat{i} + \frac{dy}{dl} \hat{j} \right) \, dl
\]

\[
\vec{F} = F_x(l) \hat{i} + F_y(l) \hat{j}
\]

\[
W = \int \vec{F} \cdot d\vec{l} = \int (F_x \frac{dx}{dl} + F_y \frac{dy}{dl}) \, dl
\]
Springs

Generally the more you stretch it, the more it pulls.

\[ F_p = kx \]

So, since \( F_s + F_p = ma \),

\[ F_s = -kx \quad F_s = |F_s| = kx \]

indicates force is opposite displacement.

If you push in

\[ F = -kx = k(-x) \quad x < 0 \]
What's the work done moving the spring to $x_f$?

$$W_p = \int_a^b F \cdot dl = \int_a^b F_p \hat{c} \cdot dx \hat{i}$$

Work done by puller

Notice

$$W_s = \int_{x_0}^{x_f} F_s \cdot dl = \int_{x_0}^{x_f} lF_s \hat{c} \cdot dx \hat{i} = -\int_0^{x_f} kx \, dx$$

Work done by spring

$$= -\frac{1}{2}kx_f^2$$
Work and Kinetic Energy (Why we care)

\[ F_p \rightarrow \begin{array}{c}
0 \\
\Delta x \\
0
\end{array} \]

Can relate the work done to the change in speed without talking about time

\[ F_p = ma \Rightarrow \frac{dv}{dt} = \frac{1}{m} \frac{dV}{dx} \]

\[ F_p = m \frac{dV}{dx} \]

\[ \int_{i}^{f} F_p \, dx = \int_{i}^{f} m \frac{dV}{dx} \, dx \]

\[ W_p = \frac{1}{2} m v_i^2 \left| \frac{f}{i} \right| = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2 \]

We call:

\[ K = \frac{1}{2} m v^2 \equiv \text{the kinetic energy} \]

So

\[ W_p = \Delta K = K_f - K_i \]
We treated one force for many forces $F_1, F_2, \ldots$ each force would do work and $W_1, W_2, \ldots$

$$W_1 + W_2 + \ldots = \Delta KE = \frac{1}{2} m v_f^2 - \frac{1}{2} m v_i^2$$

Example:

Estimate the work done when launching a bullet:

Before

After

$$W = \Delta K = K_f - K_i = \frac{1}{2} m v_f^2 - 0$$

$m = 5 \text{ g}, v_f = 500 \text{ m/s}$

$$W = \frac{1}{2} (0.005 \text{ kg}) (5 \times (5 \text{ m/s})^2) = 625 \text{ J}$$
Proof

\[ dW = \vec{F}_g \cdot d\vec{l} = mg \cos \theta \; dl = -mg \; dy \]

\[ \cos \theta \; dl = \cos (90^\circ + \theta) \; dl = -\sin \theta \; dl = -dy \]

Now

\[ W = \int_a^b dW = \int_a^b -mg \; dy = -mg (y_2 - y_1) \]

\[ W_g = -mg (y_2 - y_1) \]

So consider a closed loop

\[ W_g = -mg (y_2 - y_1) + \underbrace{-mg (y_1 - y_2)}_{\text{work done}} \]

\[ W_{\text{total}} = 0 \]