Oscillation:

\[ F = -kx \]

We will show today:

1. The motion is sinusoidal

2. The frequency is \( f = \frac{1}{2\pi} \sqrt{\frac{k}{m}} = \text{cycles/sec} \)

   the period \( T = \frac{\text{secs}}{\text{cycle}} = \frac{1}{f} \)

Analysis

\[ F = ma = -kx \]

\[ \frac{md^2x(t)}{dt^2} = -kx(t) \]
(1) This is a differential equation for \( x(t) \)

Second order \( \equiv \) two derivatives \( \frac{d^2x}{dt^2} \)

(2) It can also be written as two first order differential equations

\[
\begin{align*}
\frac{dx}{dt} &= v \\
\frac{m}{dt} \frac{dv}{dt} &= -kx
\end{align*}
\]

The goal: one for every first order equation

Given the position and velocity at time \( t_0 \), \((x_0\text{ and } v_0)\) determine the position and velocity at all subsequent times.

In the following:

\[
\frac{d^2x}{dt^2} = -\frac{k}{m} x
\]

We are motivated to try

\( x(t) = \cos \omega t \)
Then

\[ \ddot{x} = -\omega_0^2 \sin \omega_0 t \]
\[ \ddot{x} = -\omega_0^2 \cos \omega_0 t \] (it comes back)

So

\[ \ddot{x} = -\omega_0^2 \cos \omega_0 t = -\omega_0^2 x(t) \]

So \( x = \cos \omega_0 t \) is a solution to the differential equation provided \( \omega_0^2 = \frac{k}{m} \)

But this is not the only solution:

\( C_1 \cos \omega_0 t \) is a solution

\( C_2 \sin \omega_0 t \) is also a solution provided \( \omega_0^2 = \frac{k}{m} \)

The general solution (not proved)

\[ x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t \]
\[ \omega_0 = \sqrt{\frac{k}{m}} \]

- The constants \( C_1 \) and \( C_2 \) are adjusted to match the two initial conditions \( x_0, v_0 \).
Note

\[ x(0) = x_0 = C_1 \cos(\theta) + C_2 \sin(\theta) \]

\[ x_0 = C_1 \]

Now

\[ \dot{x} = -C_1 \omega_0 \sin \omega_0 t + C_2 \omega_0 \cos \omega_0 t \]

\[ v_0 = v(0) = -C_1 \omega_0 \sin \theta + C_2 \omega_0 \cos \theta \]

\[ \frac{v_0}{\omega_0} = C_2 \]

So Summary the general solution

\[ x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \]

\[ \omega_0 = \sqrt{\frac{k}{m}} \quad \text{units} \quad \text{Time} \]
Case 1 at Time $t=0$ we release the spring

$v_0 = 0$

$x = x_0 \cos \omega t$

So

- Amplitude $A = x_0$ (in this case)
- Period: $A \cos \left(\frac{2\pi t}{T}\right)$

$$\frac{2\pi}{T} = \omega_0 \Rightarrow \quad T = \frac{2\pi}{\omega_0} = 2\pi \sqrt{\frac{m}{k}}$$

Now

$$f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}$$
Case 2

At time $t_0 = 0$ we give it a kick with speed $v_0$.

\[
x = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t) = \frac{v_0}{\omega} \sin(\omega t)
\]

Then:
- $A = v_0 / \omega$
- Position = 0 at time = 0
- The slope at time = 0 is $v_0$
General Case

Expect some general oscillation

\[ A = ? \]

- Amplitude?
- Phase

\[ x(t) = x_0 \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \]

\[ A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \quad \text{will turn out to be the amplitude} \]

\[ x(t) = A \left[ \frac{x_0}{A} \cos(\omega_0 t) + \frac{v_0}{\omega_0} \sin(\omega_0 t) \right] \]

\[ x(t) = A \left[ \cos \phi \cos(\omega_0 t) + \sin \phi \sin(\omega_0 t) \right] \]
\[ \cos(A - B) = \cos A \cos B - \sin A \sin B \]

So

\[ x(t) = A \cos(\omega_0 t - \phi) \]

\[ A = \sqrt{x_0^2 + \left(\frac{v_0}{\omega_0}\right)^2} \quad \tan \phi = \frac{v_0}{\omega_0} \]

\[ \Delta t = \frac{\phi}{\omega_0} \]
Energy in SHM:

\[ E_0 = \frac{1}{2} k x_0^2 + \frac{1}{2} m v_0^2 = \text{Constant} \]

The total energy is

\[ \frac{1}{2} k x^2 + \frac{1}{2} m v^2 = E_0 \]

So at every point during the evolution

\[ \text{PE}(t) + \text{KE}(t) \]

So the picture is:

\[ U = \frac{1}{2} k x^2 \]

\[ x = -A \quad x = A \]

\[ X \quad X \quad X \quad X \quad X \]
Important Case 1

\[ x = A_{\text{max}} \]

\[ \frac{1}{2} k x_{\text{max}}^2 = \frac{1}{2} k A_{\text{max}}^2 = E_0 \]

or

\[ \frac{1}{2} k A_{\text{max}}^2 = \frac{1}{2} k x_0^2 + \frac{1}{2} m v_0^2 \]
Then

\[ A^2 = x_0^2 + \frac{m}{k} v_0^2 \quad \omega_0 = \sqrt{\frac{k}{m}} \]

So

\[ A^2 = x_0^2 + \frac{v_0^2}{\omega_0^2} \]

\[ A = \left( x_0^2 + \frac{v_0^2}{\omega_0^2} \right)^{1/2} \quad \text{agrees with before} \]

(2) Important case #2

\[ \frac{1}{2} m v_{\text{max}}^2 = E_0 = \frac{1}{2} k A^2 \]

\[ v_{\text{max}} = \sqrt{\frac{k}{m} A} \]

From before

\[ x(t) = A \cos(\omega_0 t - \phi) \]

\[ v_0 = \dot{x}(t) = -A \sin(\omega_0 t - \phi) \cdot \omega_0 \]
Hard Problem @ Todays Lecture

$V_0 = ?$

$M = 1 \text{ kg}$

$\kappa = 2 \frac{N}{m}$

$x_0 = 0.2 \text{ m}$

After a time $t = 1.56 \text{ s}$ the block crosses the origin. Determine its maximum speed and initial speed of cross.

$x(t) = A \cos(\omega t - \phi)$

Now:

$x(t=0) = A \cos(-\phi)$

$x(t_{\text{cross}}) = 0 = A \cos(\omega t_{\text{cross}} - \phi)$

So these equations become:

$x_0 = A \cos \phi$ \hspace{1cm} (1)

$\omega t_{\text{cross}} - \phi = \frac{\pi}{2}$ \hspace{1cm} (2)

$\omega_0 = \sqrt{\frac{\kappa}{m}} = \sqrt{\frac{2}{1}} = 1.414 \text{ s}$

So:

$\phi = \omega t_{\text{cross}} - \frac{\pi}{2} = 1.414 \text{ s} \cdot 1.56 \text{ s} - \frac{\pi}{2} = 0.635$

And:

$A = \frac{x_0}{\cos \phi} = 0.25 \text{ m}$
\[ \frac{1}{2} m v_{\text{max}}^2 = \frac{1}{2} k A^2 \]

\[ v_{\text{max}} = \sqrt{\frac{k}{m}} A = 0.35 \text{m/s} \]

The initial velocity

\[ x(t) = A \cos(\omega_0 t - \phi) \]

\[ v(t) = \dot{x}(t) = -A \omega_0 \sin(\omega_0 t - \phi) \]

\[ v_0 = v(t=0) = -A \omega_0 \sin(-\phi) \]

\[ = - (0.25 \text{m}) \left( \frac{1.41 \frac{1}{s}}{s} \right) \sin(-0.635) \]

\[ v_0 = 0.21 \text{ m/s} \]