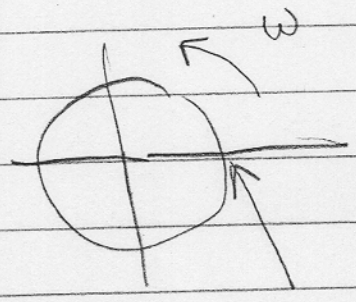
So Far



$\tau = RF_{\perp}(\pm) = RF\sin\theta\ (\pm) \qquad 0 < \theta < 180^{\circ}$

$\tau = I\alpha$

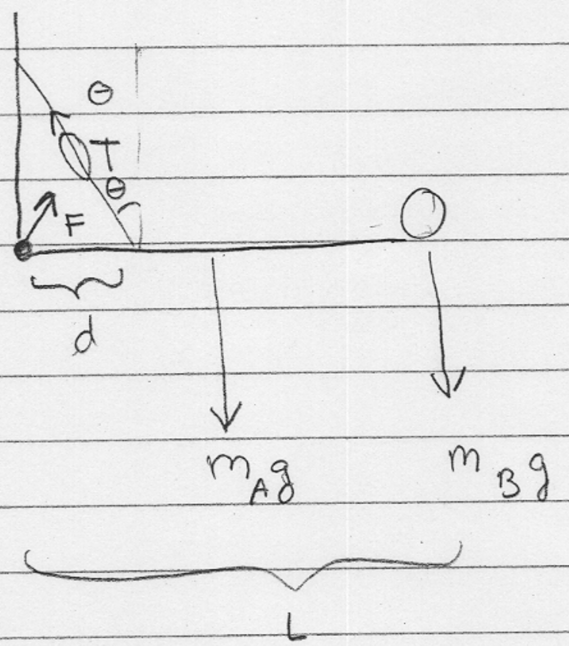
Then Today

- Statics - An applications of what we know Ch 12
- Onward to 3D, cross product and more
- Angular Momentum Conservation

1 |

Static Problems - Most of what we need from Ch 12

Example: Bicep



$L = 30\,cm$

$d = 4\,cm$

$m_a = 1.5\,kg$

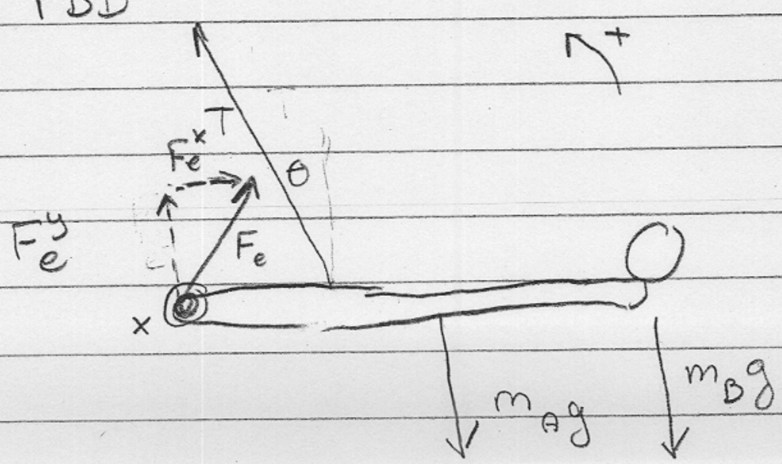
$m_B = 5\,kg$

$\theta = 10°$

→ Determine the tension and the force in the elbow

Solution

① Draw a FBD



② Write $F = m\cancel{a}^{0}$ and $\tau = I\cancel{\alpha}^{0}$ the accelerations are zero since the situation is static

→ Break up forces // and ⊥ to moment arm
→ Choose a pivot point (

$x|$

$$-T\sin\theta + F_e^x = 0$$

$y|$ $$T\cos\theta + F_e^y - m_a g - m_B g = 0$$

$\tau|$ $$\vec{\tau}_e + \vec{\tau}_T + \vec{\tau}_A + \vec{\tau}_B = 0$$

$$\star \quad 0 + T\cos\theta d\,\hat{k} + \frac{L}{2} m_a g(-\hat{k}) + L m_B g(-\hat{k}) = 0$$

③ Count unknowns and knowns and solve

Unknowns: F_e^x, F_e^y, T

From ★

$$T\cos\theta d = \frac{L}{2} m_a g + L m_B g$$

So $\quad L = 30\text{ cm} \quad d = 4\text{cm} \quad \cos\theta \approx 10^\circ \quad m_A = 1.5\text{ kg} \quad m_B = 5\text{ kg}$

$$T = \frac{L m_B g}{d\cos\theta}\left(1 + \frac{m_A}{2m_B}\right) = 437\text{ N}$$

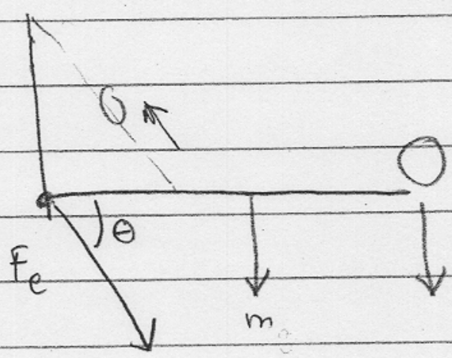
Then

$$F_e^x = T\sin\theta = m_B g \frac{L}{d}\tan\theta \left(1 + \frac{m_A}{2m_B}\right) = 76\text{ N}$$

$$F_e^y = m_A g + m_B g - \cancel{\cos\theta}\, \frac{m_B g L}{d\cancel{\cos\theta}}\left(1 + \frac{m_A}{2m_B}\right)$$

$$F_e^y = m_B g\left(1 + \frac{m_A}{m_B}\right) - m_B g \frac{L}{d}\left(1 + \frac{m_A}{2m_B}\right) = -366\text{ N}$$

Total Force on elbow



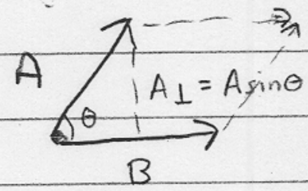
$$\sqrt{(F_e^x)^2 + (F_e^y)^2} = 373\text{ N}$$

$$\theta = \tan^{-1}\left(\frac{76\text{N}}{-366\text{N}}\right) = 12^\circ$$

Cross Product

- $\vec{\tau} = \vec{R} \times \vec{F}$ To be explained

① Right hand Rule



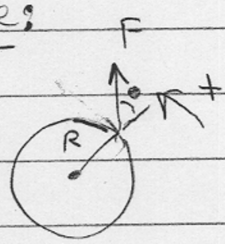
$\vec{C} = \vec{A} \times \vec{B}$

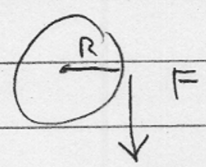
$\rightarrow |\vec{C}| = AB\sin\theta = A_\perp B$ = area of parallelogram

$\rightarrow$ Direction given by right hand rule = "an oriented area"

$\vec{A} \times \vec{B}$ points into page for this example

Example:





Torque $\vec{\tau} = \vec{r} \times \vec{F}$

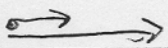
$\vec{\tau} = RF\sin\theta\ \hat{k}$ $\tau = RF\ (-\hat{k})$

out of page into page

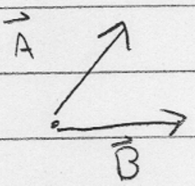
Properties

① If $\vec{A}$ and $\vec{B}$ are parallel

$$\vec{A} \times \vec{B} = 0$$



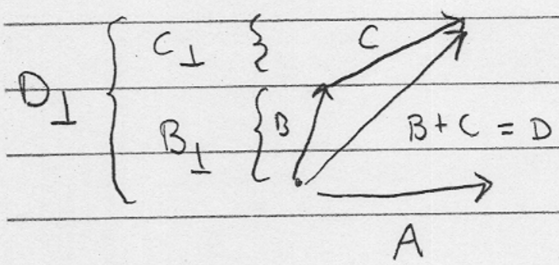
Area is zero

② 

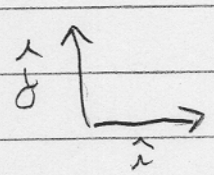
$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$$

Use the right hand rule

③ $$\vec{A} \times (\vec{B} + \vec{C}) = \vec{A} \times \vec{B} + \vec{A} \times \vec{C}$$



$$|\vec{A} \times (\vec{B} + \vec{C})| = A\overbrace{(B_\perp + C_\perp)}^{(B+\vec{C})_\perp} = A B_\perp + A C_\perp$$

④ 

$$\hat{i} \times \hat{j} = \hat{k} \qquad \hat{j} \times \hat{i} = -\hat{k} \qquad \hat{i} \times \hat{i} = \hat{j} \times \hat{j} = \vec{0}$$

Now $\vec{A} \times \vec{B}$

$$(A_x \hat{\imath} + A_y \hat{\jmath}) \times (B_x \hat{\imath} + B_y \hat{\jmath}) = A_x B_x \overset{0}{\cancel{\hat{\imath} \times \hat{\imath}}} + A_x B_y \overbrace{\hat{\imath} \times \hat{\jmath}}^{\hat{k}}$$

$$+ A_y B_x \underbrace{\hat{\jmath} \times \hat{\imath}}_{-\hat{k}} + A_y B_y \overset{0}{\cancel{\hat{\jmath} \times \hat{\jmath}}}$$

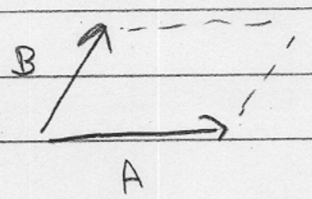
So

$$\vec{A} \times \vec{B} = (A_x B_y - A_y B_x)\, \hat{k}$$

$$= \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} \hat{k}$$

determinant

Determinants an



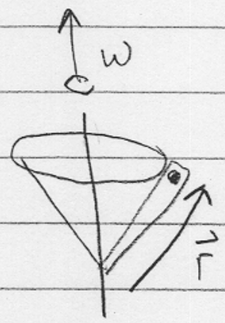
$$\begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix} = \pm \text{ area of parallelogram}$$

$$= A_x B_y - B_x A_y$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = \hat{\imath} \begin{vmatrix} A_y & A_z \\ B_y & B_z \end{vmatrix} - \hat{\jmath} \begin{vmatrix} A_x & A_z \\ B_x & B_z \end{vmatrix}$$

$$+ \hat{k} \begin{vmatrix} A_x & A_y \\ B_x & B_y \end{vmatrix}$$

Other Examples:



$\vec{v} = \vec{\omega} \times \vec{r}$ for a fixed point on the rigid body

$|v| = \omega r \sin\theta = \omega R$

direction of $\vec{v}$ into page

Around A principle axis:

$\vec{\tau} = I \dfrac{d\vec{\omega}}{dt}$ (Some corrections to this in later courses)

Angular Momentum

So

$$\sum \vec{\tau} = I \frac{d\vec{\omega}}{dt} \quad \text{analogous} \quad \sum \vec{F} = m \frac{d\vec{v}}{dt}$$

$$\sum_{ext} \tau = \frac{d(I\vec{\omega})}{dt} \quad \text{analogous} \quad \sum_{ext} \vec{F} = \frac{d(m\vec{v})}{dt}$$

if $F_{ext} = 0$ $\vec{p}$ const

Define:

points

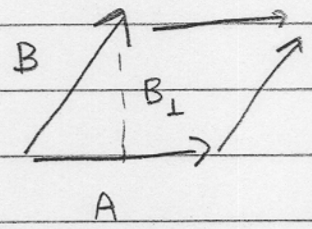
$\vec{L} = I\vec{\omega}$ the angular momentum of a rigid body along a principle axis

Note: if there are no external torques the total angular momentum is constant

$$\frac{d\vec{L}}{dt} = 0 \implies \vec{L} = \text{constant}$$

Last Time

① Cross Product

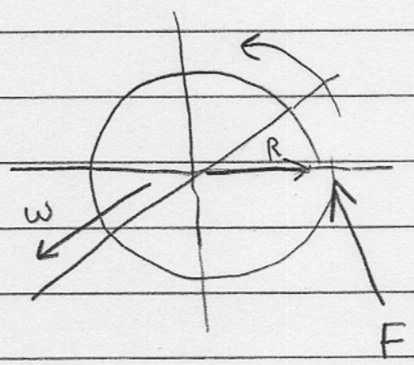


$\vec{A} \times \vec{B} \Rightarrow$ Direction given by RHR (Right

$|\vec{A} \times \vec{B}| = A_\perp B = A B_\perp = AB\sin\theta$

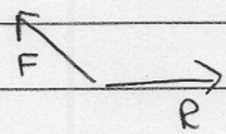
= area of Parallelogram

② Vector Nature of Rotational motion:



a) $\vec{\tau} = \vec{R} \times \vec{F}$ → points out of page

$\vec{\tau} = RF_\perp \hat{k}$



b) Causes spinning out of the page

$\vec{\omega} = \omega \hat{k}$ $\quad \vec{v} = \vec{\omega} \times \vec{r}$

c) Along a principle axis

$\sum \vec{\tau}_{ext} = I \vec{\alpha}$

(3) Then we introduced the angular moment

$\vec{L} = I\vec{\omega}$ ← true around a principle axis

Then

$\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$ analogous to $\vec{F} = \frac{d\vec{p}}{dt}$

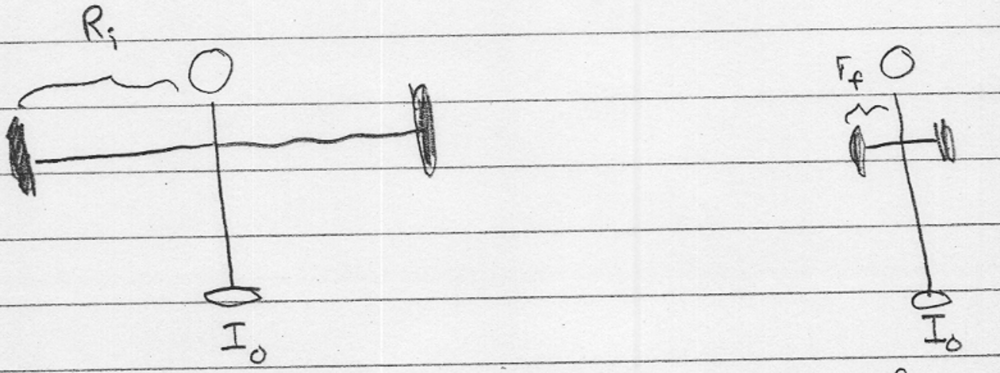
If there are no external torques

$\vec{L}_{TOT} = \text{Constant}$

So in a simple spin

$I_i \omega_i = I_f \omega_f$

Simple Model of Figure Skater -- see movie



initial final

The moment of ineritia Initial

$$I_o = \cancel{I_o} + 2mR_i^2 = 2mR_i^2$$

pretend small

The final moment of inertia

$$I_f = \cancel{I_o} \simeq 2mR_f^2 \simeq 2mR_f^2$$

Lets estimate (based on watching movie)

$$\frac{I_o}{I_f} \simeq \frac{2mR_i^2}{2mR_f^2} \simeq (2)^2 \simeq 4$$

guess based on movie

So using Angular momentum

$$I_i\omega_i = I_f\omega_f \Rightarrow \omega_f = \frac{I_i}{I_f}\omega_f$$

From watching the movie

ω_i = rotational speed just before spin $\simeq$ 60rpm

So

$$\omega_f \simeq \frac{I_i}{I_f}\omega_i \simeq 4 \cdot 60 \simeq 240 \text{ rpm}$$

about 20% off. from movie

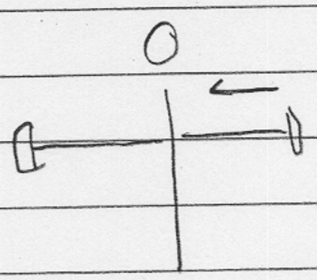
Energy is not constant:

$$K_i = \frac{1}{2} I_i \omega_i^2$$

$$K_f = \frac{1}{2} I_f \omega_f^2 = \frac{1}{2} I_f \left(\frac{I_i \omega_i}{I_f}\right)^2 = \frac{1}{2} I_i \omega_i^2 \left(\frac{I_i}{I_f}\right)$$

$$K_f \simeq K_i \cdot 4$$

The additional work is provided by skater's arms



$$F = \frac{mv^2}{r} = \frac{m(\omega r)^2}{r} = m\omega^2 r$$

ω a function of r

So

$$W = \int_{r_i}^{r_f} 2\vec{F} \cdot d\vec{r} = \int_{[r_i, r_f]} 2m\omega^2 r \, dr$$

Now

$$I_r \omega_r = I_i \omega_i \qquad I_r = 2mr^2$$

$$\omega_r = \frac{I_i \omega_i}{2mr^2}$$

So

$$W = \int_{r_f}^{r_i} 2m \left(\frac{I_i \omega_i}{2mr^2} \right)^2 r \, dr$$

$$W = \frac{I_i^2 \omega_i}{2m} \int_{r_f}^{r_i} \frac{dr}{r^3}$$

$$= \frac{I_i^2 \omega_i^2}{2m} \left(\frac{-1}{2r^2} \right) \Bigg|_{r_f}^{r_i}$$

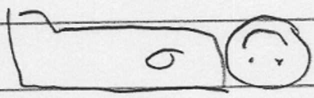
$$= \frac{1}{2} \frac{I_i^2 \omega_i^2}{2m} \left(\frac{1}{r_f^2} - \frac{1}{r_i^2} \right) \qquad I_i = 2mr_i^2$$

$$I_f = 2mr_f^2$$

$$W = \frac{1}{2} I_i \omega_i^2 \left(\frac{I_i}{I_f} - 1 \right) = K_f - K_i$$

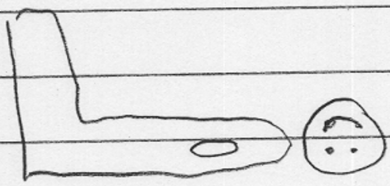
Cat Falling

Step 1



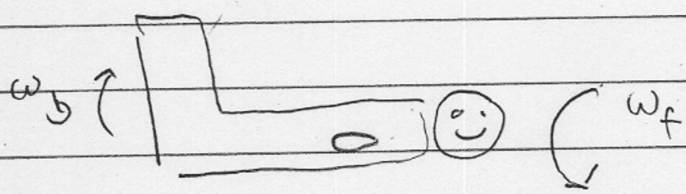
$$I_{front} \simeq I_{back}$$

Step 2



$$I_{back} \gg I_{front}$$

Step 3

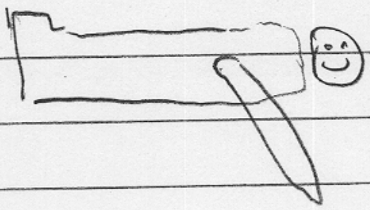


$$I_{front}\,\omega_{front} + I_{back}\,\omega_{back} = 0$$

$$-\frac{I_{front}\,\omega_{front}}{I_{back}} \simeq \omega_{back}$$

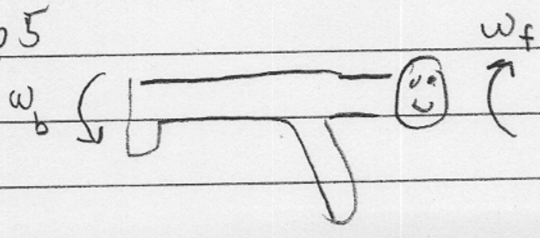
small since $I_b \gg I$

Step 4



$$I_b \ll I_f$$

Step 5

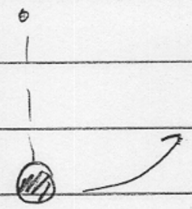


$$I_b \omega_b + I_f \omega_f = 0$$

$$\frac{I_b \omega_b}{I_f} = \omega_f$$

small since $I_f \gg I_b$

Angular Momentum of a single partice



$\ell = I\omega$ (direction out of page)

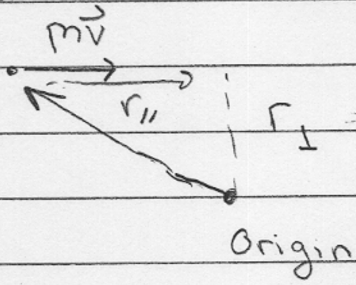
$\ell = mr^2\omega = r\, mr\omega = r\, mv$

$\ell = rp$ (direction of of page)

Motivates

$$\vec{\ell} = \vec{r} \times \vec{p}$$

Motion in a straight Line;



$|\vec{\ell}| = |r \times \vec{p}| = r_{\perp} p = r_{\perp} mv$

into page

Also a good opportunity to use vectors

$\vec{r} = r_{\parallel}\hat{\imath} + r_{\perp}\hat{\jmath}$ $\qquad \vec{p} = mv\hat{\imath}$

$\vec{\ell} = \vec{r} \times \vec{p} = (r_{\parallel}\hat{\imath} + r_{\perp}\hat{\jmath}) \times (mv\hat{\imath})$ $\qquad \hat{\imath} \times \hat{\imath} = 0$

$\hat{\jmath} \times \hat{\imath} = -\hat{k}$

$\ell = r_{\perp} mv(-\hat{k})$

Dynamics

$$\vec{F} = \frac{d\vec{p}}{dt}$$ holds in any coordinate system which moves @ constant v ≡ an inertial frame

So

$$\frac{d}{dt}(\vec{r} \times \vec{p}) = \overbrace{\frac{d\vec{r}}{dt} \times p}^{\vec{v} \times m\vec{v} = 0} + \vec{r} \times \overbrace{\frac{d\vec{p}}{dt}}^{\vec{F}}$$

$$\frac{d}{dt}(\vec{\ell}) = \vec{r} \times \vec{F}$$

Summary:

① $\sum \vec{\tau}_{ext} = \frac{d\vec{p}}{dt}$ for a single partice in an inertial frame

a frame moving @ constant velocit

② For a system of particles like a rod

$\vec{L} = \sum_i \vec{l}_i$ (True in general)

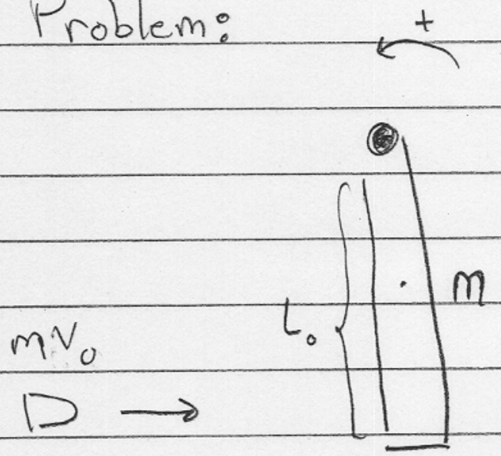
$\vec{L} = I\vec{\omega}$ (True for a rigid body rotating around the principle axis)

③ $\sum \vec{\tau}_{ext} = \frac{d\vec{L}}{dt}$ (in an inertial frame e.g. a fixed axis)

(or the center of mass in general)

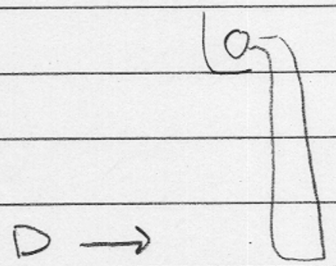
(4) If there are no external torques momentum is conserved

Problem:



Determine the bullet speed required so that the bullet + rod reaches the top of the arc

Q: Is momentum conserved? No! Suppose instead of being fastened by a pin it just rested on a ledge. Then momentum is conserved



Solution: Angular momentum is conserved ($\tau_{pin} = 0$, $\tau_{grav} = 0$

$|\vec{L}_B| = r_\perp mv = L_0 mv$ out of page or +

After impact we have:

$$L_{final} = I\omega_f$$

Now:

$$I = \underbrace{\frac{1}{3}mL_0^2}_{\substack{\text{moment} \\ \text{of intertial} \\ \text{of rod} \\ \to \text{from table}}} + \underbrace{mL_0^2}_{\substack{\text{moment of inertia} \\ \text{of bullet lodged} \\ \text{in rod}}} = \frac{4}{3}mL_0^2$$

So

$$L_{init} = L_{final}$$

$$\cancel{L_0}\cancel{m}v = \frac{4}{3}\cancel{m}L_0^{\cancel{2}}\omega_f$$

$$\boxed{\frac{3}{4}\frac{v}{L_0} = \omega_f}$$

Now if we want to reach the top of the arc

$$\frac{1}{2}I\omega^2 = mg\overbrace{\left(\frac{L_0}{2} - -\frac{L_0}{2}\right)}^{\Delta h}$$

$$\frac{1}{2}\left(\frac{4}{3}mL^2\left(\frac{3v}{4L_0}\right)^2\right) = mgL_0$$

$$\frac{3}{8}mv^2 = mgL_0 \Rightarrow \boxed{v = \sqrt{\frac{8}{3}gL_0}}$$