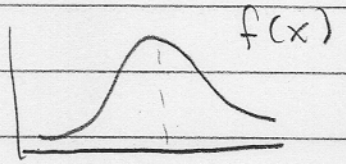
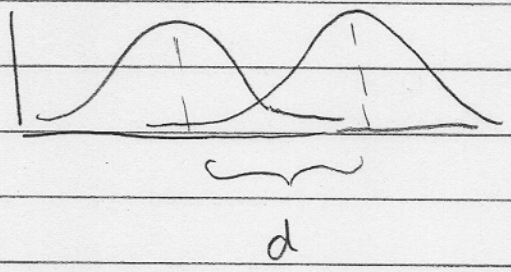
Waves

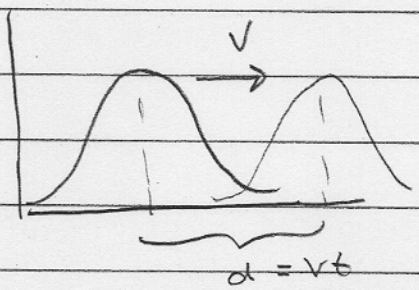
Preliminaries:





← Because when x is larger you get the function at an earlier point

What is a wave?



After a time t the wave has shifted a distance $d = vt$

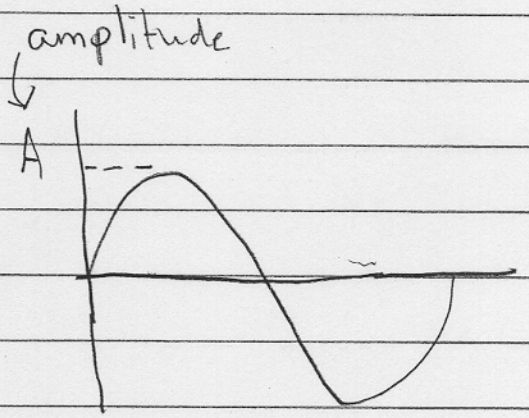
So a travelling wave has

$f(x - vt)$ ← describes a right moving wave

$f(x + vt)$ ← describes a Left moving wave

Sinusoidal Waves

$$y(x, t=0) = A \sin \frac{2\pi x}{\lambda}$$



Now we can make it move

$$y(x,t) = A \sin\left(\frac{2\pi}{\lambda}(x - vt)\right)$$

Any fixed point simply oscillates up and down with period T

$$v = \frac{\lambda}{T} \quad \text{or} \quad v = \lambda f$$

So

$$y(x,t) = A \sin\left(\frac{2\pi x}{\lambda} - \frac{2\pi}{T} t\right)$$

gets tedious to write: $\frac{2\pi x}{\lambda} - \frac{2\pi t}{T}$

$$\omega = \frac{2\pi}{T} \qquad k = \frac{2\pi}{\lambda}$$

So

$$\boxed{y(x,t) = A\sin(kx - \omega t)}$$

Notice

$$v = \lambda f = \frac{2\pi}{k}\left(\frac{\omega}{2\pi}\right) = \frac{\omega}{k} \Rightarrow \boxed{\omega = vk}$$

For a left mover

$$y(x,t) = A\sin(kx + \omega t)$$

Notation

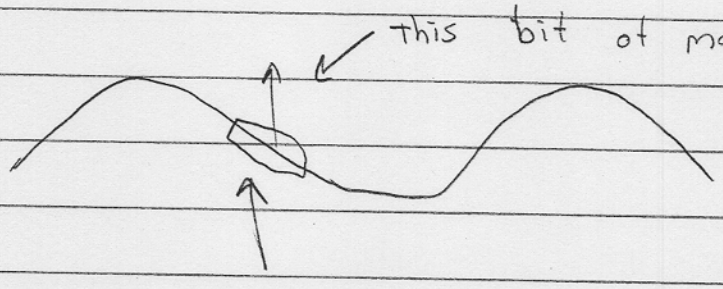
$\frac{\partial y}{\partial x} = \frac{dy}{dx}$ at a fixed time $\longleftrightarrow$ treat t as a constant

$$\frac{\partial y}{\partial x} = A\cos(kx - \omega t)\,k$$

$\frac{\partial y}{\partial t} = \frac{dy}{dt}$ at a fixed time $= A\cos(kx - \omega t)(-\omega)$

↑ treat x as constant

Then The wave equation



String stretched $\Leftrightarrow$ PE $\sim \frac{1}{2} k \Delta x^2$

Characterizing the String we have:

① $\mu = \frac{\text{mass}}{\text{Length}}$ $\quad \Delta m = \mu \Delta x$



② F_T the tension in the rope

Then



$\sum F_y = m a_y$ $\quad \sum F^x = m \ddot{x}^0$

$F_T \sin\theta_2 - F_T \sin\theta_1 = \mu \Delta x \frac{\partial^2 y}{\partial t^2}$ $\quad F_T \cos\theta_2 - F_T \cos\theta_1 = 0$

Now for small angles

$\sin\theta \simeq \tan\theta \simeq \frac{\partial y}{\partial x}$ ← slope of line at a particular time

$\cos\theta \simeq 1$

So from x equation

$F_T - F_T = 0$ learn that the tension is constant

And

$$F_T\left(\frac{\partial y}{\partial x}\right)_2 - F_T\left(\frac{\partial y}{\partial x}\right)_1 = \mu \, \Delta x \, \frac{\partial^2 y}{\partial t^2}$$

$$\frac{F_T\left[\left(\frac{\partial y}{\partial x}\right)_2 - \left(\frac{\partial y}{\partial x}\right)_1\right]}{\Delta x} = \mu \frac{\partial^2 y}{\partial t^2}$$

$$\underbrace{F_T \frac{\partial^2 y}{\partial x^2}}_{\text{Net force in the direction}} = \underbrace{\mu \frac{\partial^2 y}{\partial t^2}}_{ma}$$

Now this is normally written:

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{V^2}\frac{\partial^2 y}{\partial t^2}$$

The wave equation

with $V = \sqrt{\frac{F_T}{\mu}}$

We will show now

① This equation has wavelike solutions

② The velocity of the wave is $V = \sqrt{\frac{F_T}{\mu}}$

Now

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

Want to show that $f(x \mp vt)$ is a solution to this equation

$$y = f(x \mp vt)$$

$$\frac{\partial y}{\partial x} = f'(x-vt) = f'$$

$$\frac{\partial y}{\partial t} = f'(x-vt)(-v) = -f'v$$

$$\frac{\partial^2 y}{\partial x^2} = f''(x-vt) = f''$$

$$\frac{\partial^2 y}{\partial t^2} = f''(x-vt)(-v)(-v) = f''v^2$$

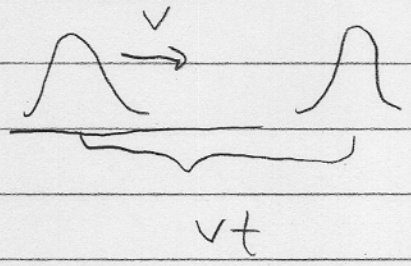
So

$$\boxed{\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}} = \frac{1}{v^2} f''v^2 = f''$$

↖ see also ex 15-6 from book

Last Times

① Waves



$f(x - vt)$ ← right mover

$f(x + vt)$ ← left mover

② Sinusoind Waves:

$y(x,t) = A\sin(kx - \omega t)$

$= A\sin(k(x - \frac{\omega}{k}t))$

$x - vt$

$k = \frac{2\pi}{\lambda}$ $\omega = \frac{2\pi}{T} = 2\pi f$

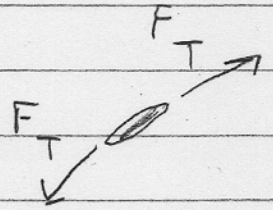
$v = \frac{\omega}{k}$

③



$\mu = \frac{\text{mass}}{\text{length}}$

F_T = tension



$F_T \frac{\partial^2 y}{\partial x^2} = \mu \frac{\partial^2 y}{\partial t^2}$

net force ma

(4) Wave equation

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2} \qquad v = \sqrt{\frac{\mu}{F_T}}$$

Showed

$$y = f(x - vt) + g(x + vt)$$

is a solution. Actually the most general solution

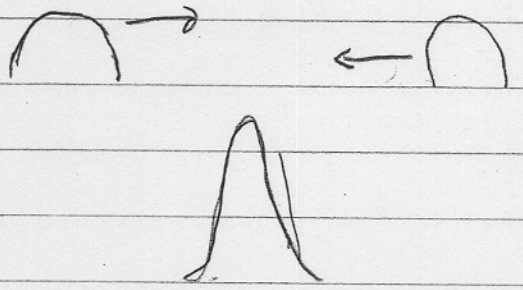
Superposition

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$

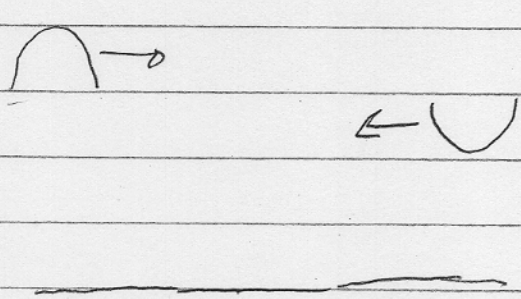
If y_1 and y_2 are solutions then so is

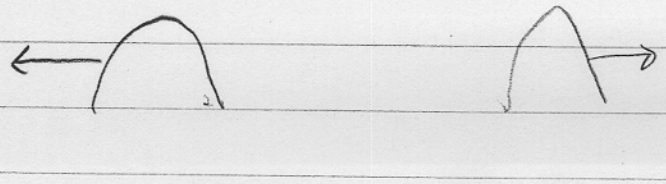
$y_{TOT} = y_1 + y_2$

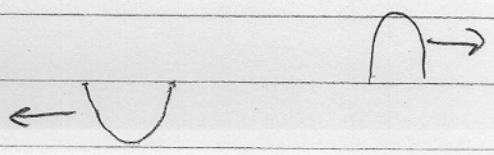
Constructive Interference:



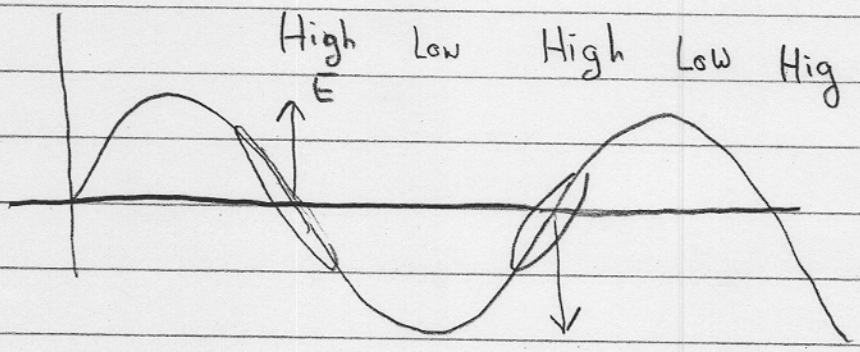
Destructive Interference:



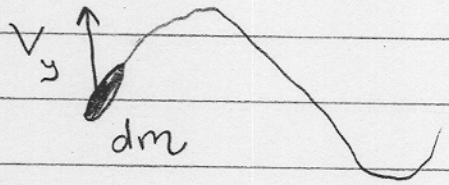




Energy in the Wave



There is kinetic energy (parts moving up and down) and potential (stretching of string)



$$dm = \mu\, dx$$

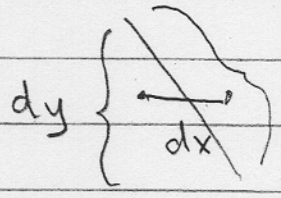
$$v_y = \frac{\partial y}{\partial t}$$

So

$$dK = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 dx$$

$$\frac{dK}{dx} = \text{kinetic energy per length} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

The potential energy



$$dl_f = \sqrt{dy^2 + dx^2}$$

$$dW_{F_T} = \int_{dx}^{dl_f} F_T \, dl = F_T \left[dl_F - dx \right]$$

$$= F_T \left(\sqrt{dy^2 + dx^2} - dx \right)$$

$$= F_T \, dx \left(\sqrt{1 + \left(\frac{dy}{dx}\right)^2} - 1 \right)$$

$$\sqrt{1 + \left(\frac{dy}{dx}\right)^2} \simeq 1 + \frac{1}{2}\left(\frac{dy}{dx}\right)^2$$

$$\rightarrow dW = F_T \, dx \left(1 + \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 - 1 \right)$$

work done to stretch string

$$\frac{dU}{dx} = \frac{F_T}{2}\left(\frac{\partial y}{\partial x}\right)^2 = \text{potential energy per length}$$

For Sinusoidal Waves

$$y = A\cos(kx - \omega t)$$

$$\frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2$$

Now

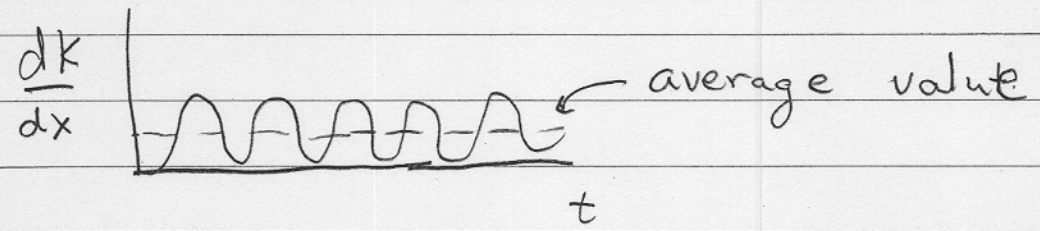
$$\frac{\partial y}{\partial t} = A\sin(kx - \omega t)\cdot\omega$$

So

$$\frac{dK}{dx} = \frac{1}{2}\mu\left(A\omega\sin(kx-\omega t)\right)^2$$

$$\frac{dK}{dx} = \frac{1}{2}\mu A^2\omega^2\sin^2(kx-\omega t)$$

Take a fixed point say $x=0$, Kinetic energy oscillates



Usually we are interested in the average kinetic energy

average ↙ ↙ averaged

$$\overline{\frac{dK}{dx}} = \frac{1}{2}\mu A^2\omega^2 \overline{\sin^2(kx-\omega t)} = \frac{1}{4}\mu A^2\omega^2$$

Now, $\overline{\sin^2(kx-\omega t)} = \frac{1}{2}$

Proof:

$$\overline{\cos^2(kx-\omega t)} = \overline{\sin^2(kx-\omega t)}$$ ← average $\sin^2$ = average $\cos^2$

But

$$\overline{\sin^2(kx-\omega t) + \cos^2(kx-\omega t)} = \overline{1}$$

$$\overline{\sin^2(kx-\omega t)} + \overline{\cos^2(kx-\omega t)} = 1$$

$$\therefore \quad \overline{\sin^2(kx-\omega t)} = \overline{\cos^2(kx-\omega t)} = \frac{1}{2}$$

Similarly for Potential Energy:

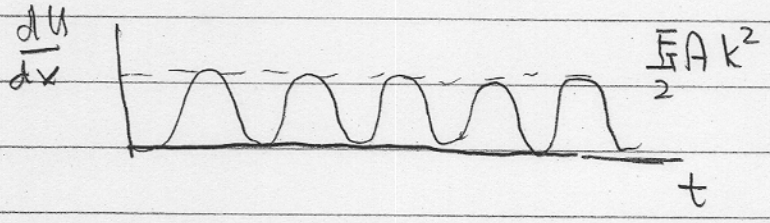
$$y = A\cos(kx - \omega t)$$

$$\frac{\partial y}{\partial x} = Ak\sin(kx - \omega t)$$

Then

$$\frac{dU}{dx} = \frac{1}{2}F_T\left(\frac{\partial y}{\partial x}\right)^2 = \frac{F_T}{2}A^2k^2\sin^2(kx - \omega t)$$

Again



$$\overline{\frac{dU}{dx}} = \frac{1}{2}\bar{F}_T A^2k^2\,\overline{\sin^2(kx - \omega t)}$$

$$\frac{d\bar{U}}{dx} = \frac{1}{4}F_T A^2k^2$$

Notice that the average potential energy is equal to the average kinetic energy

to see this:

we have

$$\left(\frac{\omega}{k}\right)^2 = v^2 \qquad v^2 = \frac{F_T}{\mu}$$

So

$$\frac{\overline{dU}}{dx} = \frac{1}{4} F_T A^2 \omega^2 \frac{k^2}{\omega^2}$$

$$\frac{\overline{dU}}{dx} = \frac{1}{4} F_T A^2 \omega^2 \left(\frac{1}{\frac{F_T}{\mu}}\right) = \frac{1}{4} \mu A^2 \omega^2 = \frac{\overline{dK}}{dx}$$

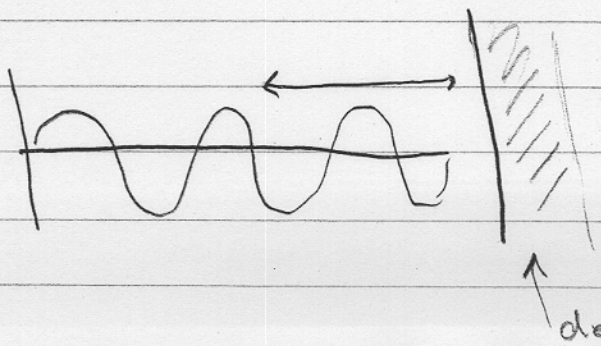
The total Energy is:

$\overline{U_E}$ ≡ average energy (in wave) per unit length

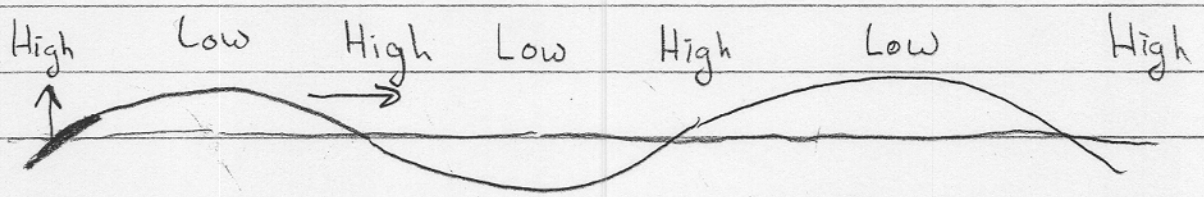
$$u_E = \frac{\overline{dK}}{dx} + \frac{\overline{dU}}{dx} = \frac{1}{4}\mu A\omega^2 + \frac{1}{2}\mu A\omega^2 = \boxed{\frac{1}{2}\mu A^2\omega^2 = u_E}$$

The ^(average) Energy Transported per unit time $= \bar{P}$

↑ the power transp in the wave



Last Time

① 

$$KE \Rightarrow \quad \frac{dK}{dx} = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 \quad \leftarrow \text{Kinetic Energy / length}$$

$$PE \Rightarrow \quad \frac{dU}{dx} = \frac{F_T}{2}\left(\frac{\partial y}{\partial x}\right)^2 \quad \leftarrow \text{Pot E per length}$$

Sinusoidal Waves

$$y = A\cos(kx - \omega t)$$

$$\frac{\partial y}{\partial t} = +A\omega \sin(kx - \omega t)$$

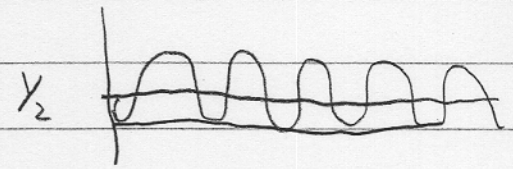
$$\frac{\partial y}{\partial x} = -Ak \sin(kx - \omega t)$$

KE

$$\frac{dK}{dx} = \frac{1}{2}\mu A^2\omega^2 \sin^2(kx - \omega t)$$

$$\overline{\frac{dK}{dx}} = \text{time averaged KE/length} = \frac{1}{2}\mu A^2\omega^2 \overline{\sin^2(kx - \omega t)}$$

$\sin^2(\omega t)$



$$\overline{\frac{dK}{dx}} = \frac{1}{2}\mu A^2\omega^2 \cdot \frac{1}{2} = \frac{1}{4}\mu A^2\omega^2$$

PE

$$\frac{dU}{dx} = \frac{F_T}{2} A^2k^2 \sin^2(kx-\omega t)$$

$$\overline{\frac{dU}{dx}} = \frac{F_T}{2} A^2k^2 \overline{\sin^2(kx-\omega t)} = \frac{F_T}{4} A^2k^2$$

Now with $\omega = vk$ $v^2 = \frac{F_T}{\mu}$

$$\overline{\frac{dU}{dx}} = \frac{F_T}{4} A^2\omega^2 \frac{k^2}{\omega^2} = \frac{F_T}{4} A^2\omega^2 \frac{1}{v^2} = \frac{F_T}{4} A^2\omega^2 \frac{1}{\frac{F_T}{\mu}}$$

$$\overline{\frac{dU}{dx}} = \frac{1}{4}\mu A^2\omega^2 = \overline{\frac{dK}{dx}}$$

↖ Average kinetic and potential equal

The total power transmitted

General Wave

u_E = Total Energy per length

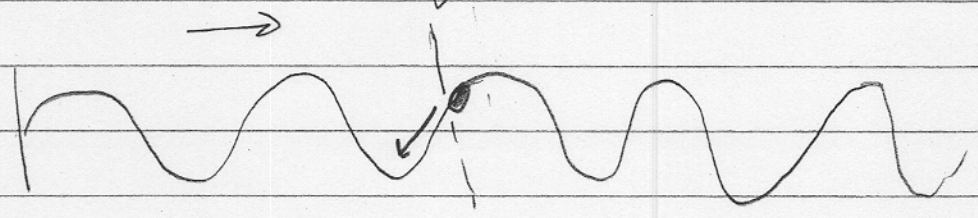
$$u_E = \frac{dK}{dx} + \frac{dU}{dx}$$

$$u_E = \frac{1}{2}\mu\left(\frac{\partial y}{\partial t}\right)^2 + \frac{1}{2}F_T\left(\frac{\partial y}{\partial x}\right)^2$$

Time averaged sinusoidal wave

$$\overline{u_E} = \frac{1}{4}\mu A^2\omega^2 + \frac{1}{4}\mu A^2\omega^2 = \frac{1}{2}\mu A^2\omega^2 = u_E$$

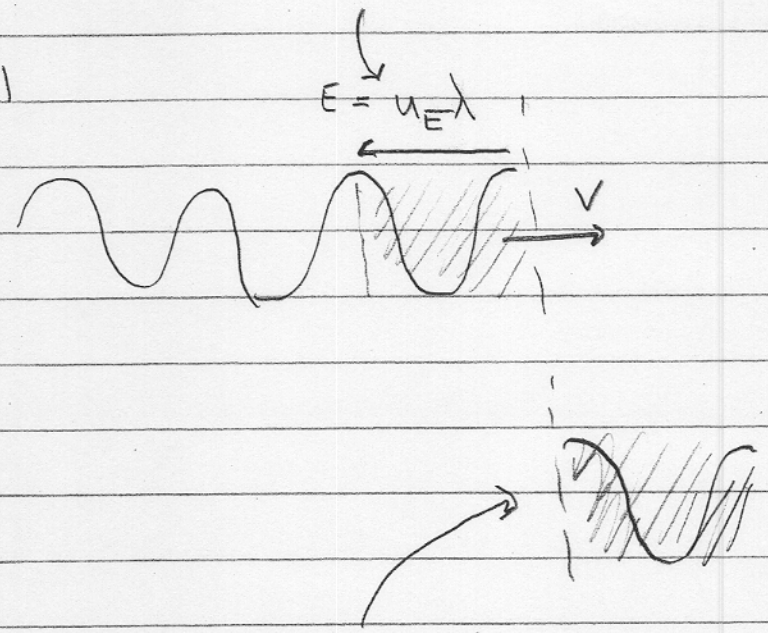
The screeen (e.g. f



Power = the amount of energy which crosses screen per unit time

$$P = -F_T \frac{\partial y}{\partial x}\frac{\partial y}{\partial t} \leftarrow \text{general wave (Homework)}$$

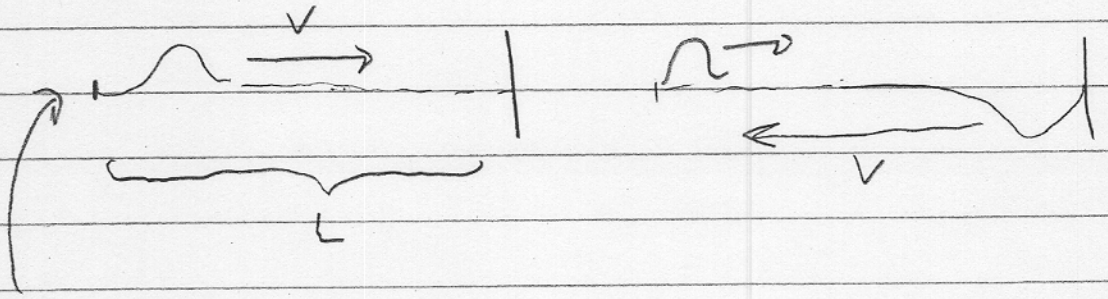
energy in one wave length



It takes one period for one wavelength to cross the detector

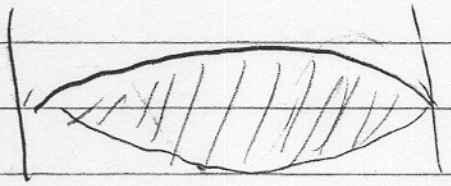
$$P = \frac{E}{T} = u_E \frac{\lambda}{T} = u_E \lambda f = \boxed{u_E V = P}$$

Resonance:

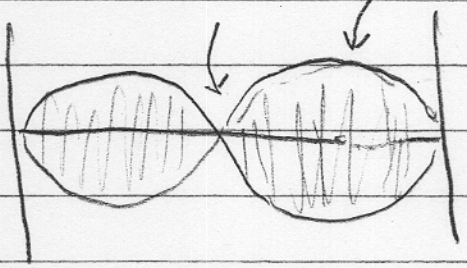


shake this end with a certain frequency, if the outgoing pulse is in phase with the returning puke the a resonance is built ap and you get a strong amplitude

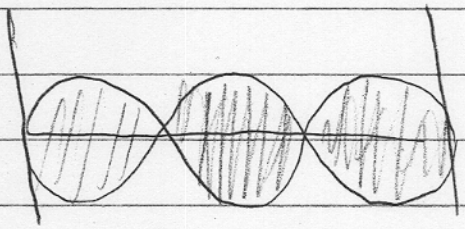
Find certain standing waves



$$L = \frac{\lambda}{2}_n$$



$$L = \lambda_n$$



$$L = \frac{3}{2}\lambda_n$$

$n = 1, 2, 3, \ldots$

In general

$$L = \frac{n\lambda_n}{2}$$

$$v = \lambda_n f_n$$

$$\frac{v}{\lambda_n} = f_n$$

$$\frac{1}{\lambda_n} = \frac{n}{2L}$$

$$\frac{nv}{2L} = f_n \qquad n = 1, 2, 3, \text{ etc}$$

Mathematical Representation of a Standing Wave

- The standing wave is a superposition of a right mover and Left mover

$$y_1 = A\sin(kx - \omega t) \leftarrow \text{right mover}$$

$$y_2 = A\sin(kx + \omega t) \leftarrow \text{left mover}$$

$$y_{TOT} = A\left[\sin(kx-\omega t) + \sin(kx+\omega t)\right]$$

Now

$$\sin(kx \pm \omega t) = \sin(kx)\cos(\pm\omega t) + \cos(kx)\sin(\pm\omega t)$$

$$\sin(kx \pm \omega t) = \sin(kx)\cos\omega t \pm \underbrace{\cos kx \sin(\omega t)}_{\text{this cancels in sum}}$$

So

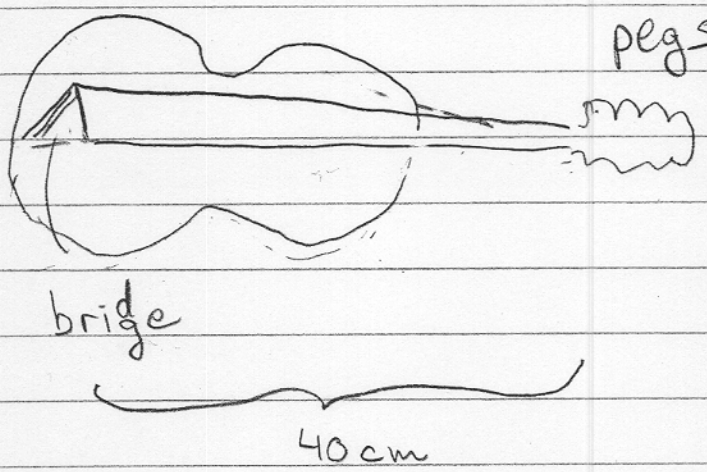
$$\boxed{y_{TOT} = 2A\sin(kx)\cos(\omega t)}$$

↑ Formula for the wave form of a standing wave

HW: Work out the KE, PE, & total E in this kind of wave

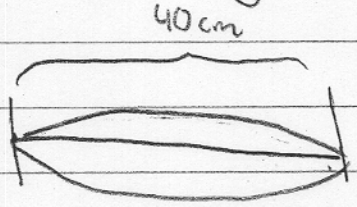
Strings

Violin



- $m_{st} \sim 0.4g \qquad \mu = \frac{1g}{40cm} \simeq 0.001\ \frac{kg}{m}$

Want the A string to resonate at A 440Hz?



- $\lambda = 80cm = 2L$
- $\frac{nv}{2L} = f_n$ from before

$$v = f_1 \cdot 2L$$

$$v = 440Hz \cdot 80cm$$
$$= 352m/s$$

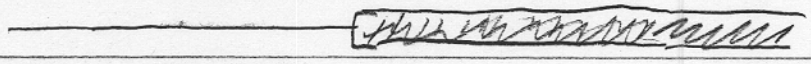
← doesn't have to be the speed of sound

Now

$$\sqrt{\frac{F_T}{\mu}} = v \qquad v^2\mu = F_T$$

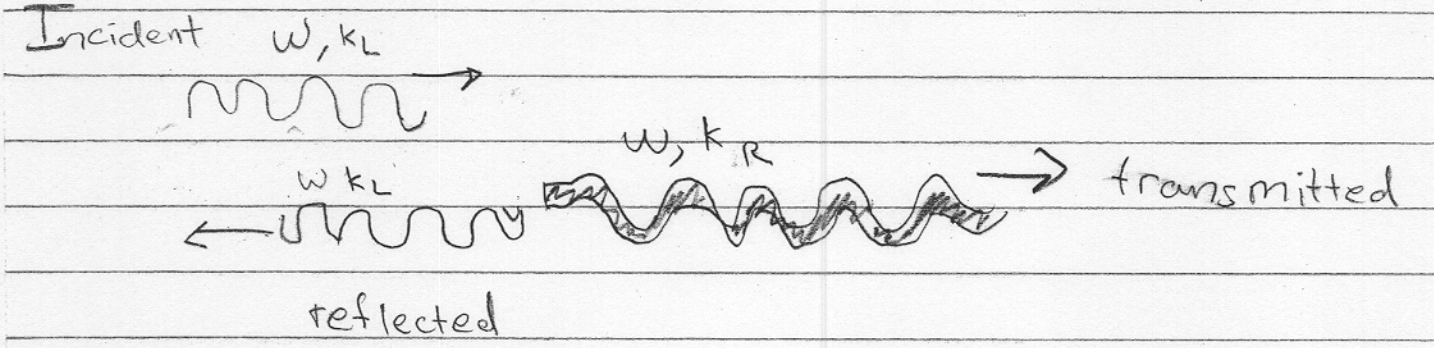
$$123\,N = F_T \quad \text{or} \quad 28\ lbs \sim$$ Best Strings ≈ 60$

Transmission and Reflection



μ_L M_R Tension is Constant

V_L V_R



The solution on the left hand side

$$y_L = A\cos(k_L x - \omega t) + B\cos(-k_L x - \omega t)$$

↖ incident wave

↖ reflected wave (write like this: $-kx-\omega t$ for later convenience

$$y_R = C\cos(k_R x - \omega t)$$

↖ transmitted wave

Want to solve for B and C in terms of A

Require:

$$y_L\Big|_{x=0} = y_R\Big|_{x=0} \quad \text{for all times} \quad \text{(no breaks)}$$

$$\frac{\partial y_L}{\partial x}\Big|_{x=0} = \frac{\partial y_R}{\partial x}\Big|_{x=0} \quad \text{for all times (no kinks)}$$

So the first condition gives

$$A\cos(-\omega t) + B\cos(-\omega t) = C\cos(-\omega t)$$

$$A + B = C$$

The second condition

$$\frac{\partial y_L}{\partial x}\Big|_{x=0} = -Ak_L\sin(k_L x - \omega t) + Bk_L\sin(-k_L x - \omega t)\Big|_{x=0}$$

$$\frac{\partial y_L}{\partial x}\Big|_{x=0} = -\sin(-\omega t)\left[Ak_L - Bk_L\right]$$

$$\frac{\partial y_R}{\partial x}\Big|_{x=0} = -\sin(-\omega t)\left[Ck_R\right]$$

Then the second Condition becomes

$$A k_L - B k_L = C k_R$$

Using the first equation

$$A k_L - B k_L = (A+B) k_R$$

$$A(k_L - k_R) = (k_L + k_R) B$$

$$A\left[\frac{(k_L - k_R)}{(k_L + k_R)}\right] = B$$

Then

$$C = A + B = A\left[1 + \frac{k_L - k_R}{k_L + k_R}\right]$$

$$= A\left[\frac{2k_L}{k_L + k_R}\right]$$

Note $k_L = \frac{\omega}{v_L} = \omega\sqrt{\frac{\mu_L}{F_T}}$ so we have $k_{L/R} \propto \sqrt{\mu_{L/R}}$

So: $$\boxed{B = A\left[\frac{\sqrt{\mu_L} - \sqrt{\mu_R}}{\sqrt{\mu_L} + \sqrt{\mu_R}}\right]}$$

$$\boxed{C = A\left[\frac{2\sqrt{\mu_L}}{\sqrt{\mu_L} + \sqrt{\mu_R}}\right]}$$