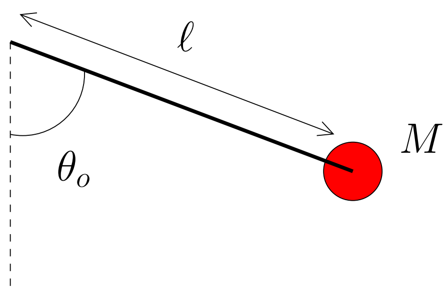
Introduction to Numerical Analysis of a pendulum

1 Introduction

The purpose of this lab is to learn about oscillation, dimensional analysis, and the utmost basics of numerical techniques.

2 Theoretical Analysis

Consider a simple mass at the end of a string. At a given initial time t_o we release the pendulum from an angle θ_o.



We showed in class that when the angle θ_o is small and friction is neglected, the motion is sinusoidal and is given by

$$\theta(t) = \theta_o \cos(\omega_o t) \qquad \omega_o \equiv \sqrt{\frac{g}{\ell}} \tag{1}$$

However when the angle is not small, this formula is no longer valid. The first goal is to determined the period of the oscillation when the initial angle is 90^o. We will also consider a small friction force on the pendulum.

$$F_D = -b\,v \tag{2}$$

1. Start by using Newton laws to determine a differential equation for the angle θ. You should find

   $$\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin(\theta) - \frac{b}{M}\frac{d\theta}{dt}$$

   (Hint: you can use simple geometry to note that $dx_{\|} = \ell d\theta$) We will write this second order differential equation as a system of first order differential equations by defining the angular velocity, $\Omega \equiv d\theta/dt$

   $$\begin{aligned} \frac{d\theta}{dt} &= \Omega \\ \frac{d\Omega}{dt} &= -\frac{g}{\ell}\sin(\theta) - \frac{b}{M}\Omega \end{aligned}$$

   with the initial condition $\theta(t=0) = \theta_o$ and $\Omega(t=0) = 0$.

2. Now introduce a set of dimensionless variables to describe the motion of the pendulum. Agree to measure mass in units of M, length in units of ℓ, and acceleration in units of g. With these three quantities we can measure all units.

   Determine a dimensionless time, small angle oscillation frequency, angular velocity, tension, kinetic energy, potential energy, and drag coefficient in this system of units:

$$\bar{t}, \bar{\omega}_o, \bar{\Omega}, \bar{T}, \bar{K}, \bar{U}, \bar{b} \tag{3}$$

3. Show that the equations can be written

$$\begin{aligned} \frac{d\theta}{d\bar{t}} &= \bar{\Omega} \\ \frac{d\bar{\Omega}}{d\bar{t}} &= -\sin(\theta) - \bar{b}\Omega \end{aligned}$$

4. Use dimensional analysis to argue that the angle as a function of time period must have the form

$$\theta(t) = F\left(\theta_o, \omega_o t, \frac{b}{M\omega_o}\right) , \tag{4}$$

   and that period of oscillation for zero damping case can be written

$$\tau_{\rm osc} = \sqrt{\frac{\ell}{g}} F(\theta_o) \, . \tag{5}$$

5. Show that the work done by friction in a time interval dt is

$$dW_{\rm fr} = -bl^2\Omega^2 \, dt \, . \tag{6}$$

   or

$$d\bar{W}_{\rm fr} = \frac{dW_{\rm fr}}{mg\ell} = -\bar{b}\bar{\Omega}^2 \, d\bar{t} \, . \tag{7}$$

3 Numerical Analysis

3.1 No friction

1. Start from an initial angle $\theta_o = \pi/2$ (in radians!). Write a program to determine how the angle θ depends on $\omega_o t$. Make a well labeled graph of the angle versus time. Also plot the small angle result $\theta(t) = \frac{\pi}{2}\cos(\omega_o t)$ for comparison. (Can you give a qualitative explanation for why the period is longer in the large angle case relative to the small angle formula)

   Determine the oscillation period. The maxima and minima of the angle as a function of time can be found by determining where $d\theta/dt$ crosses zero. This can be done by monitoring Ω and checking if it changes sign.

The period can be expressed in terms of something called an elliptic integrals of the first kind $\mathcal{F}(\phi, m)$

$$\tau_{\rm osc} = \sqrt{\frac{\ell}{g}} 4\,\mathcal{F}\left(\frac{\pi}{2}, \frac{1}{2}\right) \approx \sqrt{\frac{\ell}{g}}\, 7.4162987092054876737\,, \tag{8}$$

which is about 18% larger than the 2π result. Compare your numerical answer for the period to this value. Indicate the numerically determined oscillation period and the percent deviation from the exact number just below your graph.

2. Make a single graph of the kinetic, potential, and total energies in units of $mg\ell$ as a function of $\omega_o t$ for two values of the step size $\omega_o \Delta t = 0.01$ and $\omega_o \Delta t = 0.001$. In each case determine the percent change in the total energy of your numerical solution. This can be done by printing out the total energy at the initial step and printing out the total energy in the final step. Indicate the initial energy and the final energy in units of $mg\ell$ just below your graph. Also indicate the percent change in energy.

3.2 With friction

For simplicity take $\bar{b} = 0.25$ and run your simulation from $\omega_o t = 0 \ldots 10$. Take a step size of $\omega_o \Delta t = 0.001$

1. Make a graph of the angle as a function of time from $\omega_o t = 0 \ldots 10$ starting from $\theta_o = \pi/2$.

2. Verify the work kinetic energy theorem numerically. This theorem states that

$$W_{\rm fr} = \Delta K + \Delta U\,, \tag{9}$$

Since the initial energy is $K_i + U_i = mg\ell$ the theorem reads

$$mg\ell = K_f + U_f - W_{\rm fr}\,, \tag{10}$$

in this case. The work done by friction is

$$W_{\rm fr}(t) = \int_0^t dW_{\rm fr}\,.$$

One way to compute this is the following: As the code runs, keep a running total of the work done by friction up until that point, and store the result in an array for subsequent plotting. See `simple.pdf` for an example.

Make a graph showing as function of time the kinetic energy, the potential energy, (minus) the work by friction, and the sum of these quantities difference

$$\frac{K}{mg\ell}, \frac{U}{mg\ell}, \frac{-W_{\rm fr}(t)}{mg\ell}, \frac{K_f(t) + U_f(t) - W_{\rm fr}(t)}{mg\ell}\,. \tag{11}$$

Determine the percent change, between the initial (potential) energy and the final value of $K_f + U_f - W(t_f)$ in units of $mg\ell$. Indicate in units of $mg\ell$ the initial energy, the final value of $K_f + U_f - W_{\rm fr}(t_f)$ and the percent change just below your graph.

4 Your report

Your report should consist of

- The theoretical analyses of Sect. 2 (hand written is fine).
- Five well labeled graphs (perhaps one per page). Be sure that all axes are labeled, the value of b is reported, and the step size is given, all in appropriate scaled units. Be sure to label each curve. In the caption of each graph, answer any of the questions which were asked. It is totally fine to print out the graph and write all the labels and captions neatly by hand.
- A printout of the computer code use to generate your results. Your code should be well documented and fairly easy to read. See for example `simple.pdf`

5 Getting Started

To get started, I would recommend you download and look at the files:

1. `note.pdf` which describes the damped harmonic oscillator and dimensional analysis.
2. `simple.pdf` which is the complete numerical code I wrote for the damped harmonic oscillator.
3. A simple code for the undamped oscillator is available `simple_example.m` for download you could modify this code for your needs.
4. To get started with `matlab` you can look at `example.m`, the note `matlab.pdf`, and online.

You don't have to use `matlab` but I believe it will be significantly easier than the alternatives (C, Fortran, Mathematica, Maple, C++, Java, Python), with Python a reasonably close second.