① Recitation 10/30

Solution

For a damped oscillator:

$$x(t) = A\cos(\omega t - \phi)\, e^{-\frac{b}{2m}t}$$

$$v(t) = A\left[-\sin(\omega t - \phi)\,\omega\right] e^{-\frac{b}{2m}t}$$

$$+ A\left[\cos(\omega t - \phi)\right] e^{-\frac{b}{2m}t}\left(-\frac{b}{2m}\right)$$

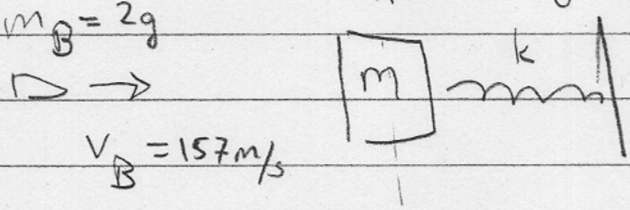
$$v(t) = A\, e^{-b/2m}\left[-\omega\sin(\omega t - \phi) - \frac{b}{2m}\cos(\omega t - \phi)\right]$$

Problem:

$x = 0$

$m = 0.5\,kg$

$m_B = 2g$



$V_B = 157\,m/s$

$m_{TOT} = M + m_B$

① (No damping) Determine the maximum compression of the spring 5pnts

(No damping)

② ↳ How far from equilibrimum is the bullet block combo after a time of $\frac{3}{8}$ of an oscillation period after impact. Draw a graph 5pnts

6pnts

③ Suppose a fairly strong damping

$$\frac{b}{2M_{TOT}\,\omega} = 0.3 \leftarrow$$

This combination $\frac{b}{m\omega_0}$ is dimensionless.

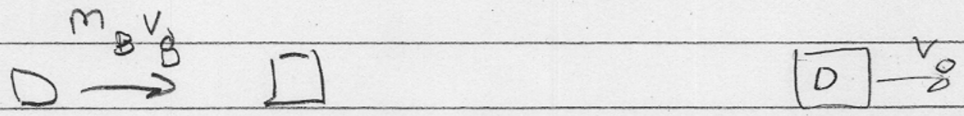
(2)

What is the position versus time? 5pnts)

(4) What is maximum extent? draw a graph?
(Not graded)

Solution

(1) From momentum conservation



$$m_B v_B = (M + m_B) v_0$$

$$\boxed{\frac{m_B}{M_{TOT}} v_B = V_0}$$

Using energy coservation:

$$\frac{1}{2} M_{TOT} V_0^2 = \frac{1}{2} k x_{max}^2$$

$$\boxed{x_{max} = \sqrt{\frac{M_{TOT}}{k}}\, V_0} = \frac{V_0}{\omega_0}$$

$= 0.495\,m$

(2) For an undamped spring

$$x(t) = C_1 \cos \omega_0 t + C_2 \sin \omega_0 t$$

C_1 and C_2 are adjusted to reproduce the initial position $x_0 = x(t=0)$ and the initial velocity

$$V_0 = \frac{dx}{dt}\Big|_{t=0}$$

③

Find

at time $t=0$ the bullet-block combo is not displaced

$$x(t) = \cancel{x_0} \cos \omega_0 t + \frac{v_0}{\omega_0} \sin \omega_0 t$$

$$x(t) = \frac{v_0}{\omega_0} \sin \omega_0 t \qquad \omega_0 = \sqrt{\frac{k}{m_{TOT}}}$$

At a time $t = \frac{3}{8} T_{period}$ $\qquad V_0 = \frac{m_B V_B}{m_{TOT}}$

$$T_{period} = \frac{2\pi}{\omega_0}$$

So $\omega_0 t = \cancel{\omega_0} \left(\frac{3}{8} \frac{2\pi}{\cancel{\omega_0}} \right) = \frac{6\pi}{8}$

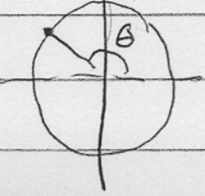
So

$$x\left(t = \frac{3}{8} T\right) = \frac{v_0}{\omega_0} \sin\left(\frac{6\pi}{8}\right)$$

$$\boxed{\omega_0 = \sqrt{\frac{k}{m_{TOT}}} \qquad V_0 = \frac{m_B V_B}{m_{TOT}}}$$

Incidentally

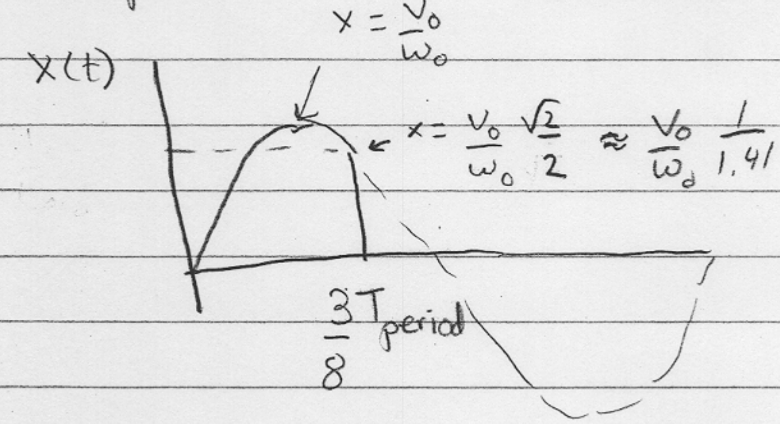
$$\sin \frac{6\pi}{8} = \sin\left(\frac{3 \cdot 2\pi}{8}\right) = \frac{\sqrt{2}}{2}$$



$$\boxed{x(t) = v_0/\omega_0 \cdot \frac{\sqrt{2}}{2}} \longrightarrow x(t) = x_{max}/\sqrt{2} = 0.3500$$

4

Graph:



3 For the damped case

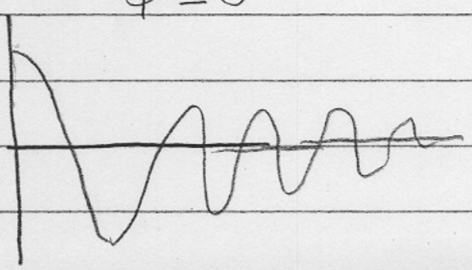
$$x(t) = A e^{-\frac{bt}{2m}} \cos(\omega t - \phi) \qquad \omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

A and ϕ are adjusted to reprodue the initial position and velocity. Since ϕ is ultimately adjusted to reproduce the initial velocity, we can equally well write

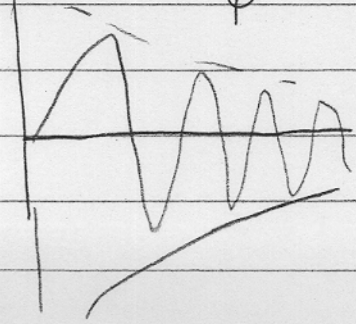
$$x(t) = A e^{-\frac{bt}{2m}} \sin(\omega t - \bar{\phi}) \qquad \bar{\phi} = \phi - \frac{\pi}{2}$$

For example

$\phi \simeq 0$



$\bar{\phi} = 0$



5

Now we have

$$x(t=0) = x_0 = 0$$

$$v(t=0) = v_0$$

From

$$x(0) = 0 = A\sin(\omega 0 - \phi)\, e^{0} \quad \text{we have } \phi = 0$$

$$\boxed{x(t) = A e^{-\frac{b}{2m}t} \sin(\omega t)}$$

$$\frac{dx}{dt} = v(t) = A e^{-\frac{b}{2m}t} \cos\omega t \cdot \omega + A e^{-\frac{b}{2m}t}\left(-\frac{b}{2m}\right) \sin\omega t$$

$$\boxed{v(t) = A e^{-\frac{b}{2m}t}\left[\omega\cos\omega t - \frac{b}{2m}\sin\omega t\right]}$$

$$v(t=0) = v_0 = A\,\omega\cos(0)$$

$$\frac{v_0}{\omega} = A \qquad \text{with } \omega = \sqrt{\omega_0^2 - \frac{b^2}{4m^2}}$$

$$\frac{v_0/\omega_0}{\sqrt{1 - \frac{b^2}{4m^2\omega_0^2}}} = A \qquad \omega = \omega_0\left(1 - \frac{b^2}{4m\omega_0^2}\right)^{\frac{1}{2}}$$

(6)

So

Now $\sqrt{1 - \frac{b^2}{4m^2\omega_0^2}} \simeq \sqrt{1-(0.3)^2} \simeq \boxed{0.95 \equiv C}$

$\omega = 0.95\omega_0 \qquad \omega = C\omega_0$

So $\qquad \omega_0 = 12.49$

$$x(t) = \frac{V_0/\omega_0}{C} e^{-0.3\omega_0 t} \sin(\overbrace{C\omega_0 t}^{11.99\frac{1}{s}})$$

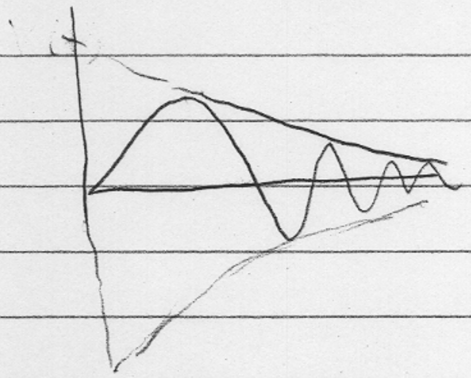
$\omega_0 = \sqrt{\frac{k}{M_{TOT}}}$

(4)

$V_0 = \frac{m_B}{M_{TOT}} V_B$

To find the maximum we want $\frac{dx}{dt} = 0$

$\frac{b}{2m\omega_0} = 0.3$



Setting $V = 0$ to find the time t_* when max is reached:

$$v(t) = Ae^{-\frac{b}{2m}t}\left[\omega\cos\omega t_* - \frac{b}{2m}\sin\omega t_*\right] = 0$$

$$\cos\omega t_* = \frac{b}{2m\omega}\sin\omega t_*$$

$$\cot(\omega t_*) = \frac{b}{2m\omega_0}\frac{1}{C}$$

7

So

$$\omega t_* = \cot^{-1} \frac{0.3}{0.95} = 1.26$$

Then

$$x = \frac{v_0/\omega_0}{C} e^{-\frac{b}{2m\omega_0} \frac{\omega_0}{\omega} \omega t} \sin(\omega t)$$

$$x^b_{max} = \frac{v_0/\omega_0}{C} e^{-\frac{0.3}{C} \cdot (\omega t_*)} \sin(\omega t_*)$$

$$x^b_{max} = \frac{x^{b=0}_{max}}{0.95} e^{-0.3 \frac{1}{0.95} 1.265} \sin(1.265)$$

$$x^b_{max} = x^{b=0}_{max} \, 0.673$$

$$x^b_{max} = 0.0333 \text{ m}$$

$$x^b_{max} = 3.3 \text{ cm}$$