

# Numerical Treatment of the Damped Oscillator

Prof. Derek Teaney

October 26, 2009

## 1 Damped Oscillator

The equation of motion is

$$M\ddot{x} + b\dot{x} + kx = 0. \quad (1)$$

We showed in class (see Appendix. A) that the general solution to these equations is

$$x(t) = A_o e^{-\frac{b}{2m}t} \cos(\omega t - \phi), \quad (2)$$

where

$$\omega = \sqrt{\omega_o^2 - \frac{b^2}{4M^2}}, \quad \text{with} \quad \omega_o = \sqrt{\frac{k}{M}}. \quad (3)$$

The amplitude  $A_o$  and the phase  $\phi$  should be adjusted to reproduce the initial position and the initial velocity. Clearly something happens when  $\frac{b}{2M}$  becomes larger than  $\omega_o$  we will discuss this later in Sect. 4.2. The analytic solution presented here is no longer valid for  $b/2M > \omega_o$

Example:

- Suppose at some time  $t = 0$ , we start the block displaced from equilibrium by a distance  $x_o$  but with no initial velocity. From the solution to the damped oscillator

$$x_o = x(0) = A_o \cos(-\phi) = A_o \cos(\phi). \quad (4)$$

To determine the initial velocity we differentiate  $x(t)$

$$v(t) = \dot{x}(t) = A_o e^{-\frac{b}{2m}t} \left[ -\frac{b}{2m} \cos(\omega t - \phi) - \omega \sin(\omega t - \phi) \right], \quad (5)$$

Then since the initial velocity is zero we have,

$$v_o = v(0) = 0 = A_o \left[ -\frac{b}{2m} \cos(\phi) + \omega \sin(\phi) \right]. \quad (6)$$

Solving these equations for  $A$  and  $\phi$  we have

$$\tan \phi = \frac{b}{2M\omega}, \quad \text{and} \quad A_o = \frac{x_o}{\cos \phi}. \quad (7)$$

The complete solution in this case is

$$x(t) = x_o e^{-\frac{b}{2M}t} \frac{\cos(\omega t - \phi)}{\cos \phi}. \quad (8)$$

Sect. 4.2 will compare this solution to a numerical treatment of the differential equation Eq. 1.

## 2 Dimensional Analysis of a Damped Oscillator

Much about what happens as a function time can be determined from a dimensional analysis of the damped oscillator. We will concentrate on the example problem given above, and show how you can almost guess the form of the solution in Eq. 8

The dimensional *constants* in the initial state are

$$[M] = kg \quad [k] = \frac{N}{m} = \frac{kg}{s^2} \quad [b] = \frac{N}{m/s} = \frac{kg}{s} \quad [x_0] = m \quad (9)$$

We now wish to write down the functional form of the position as a function of time. Since the position has units of meters it must be proportional to a constant of dimension meters. An available constant (the only actually) is  $x_o$ . Then it could be a function of the dimensionless variables which can be formed out of the constants given above and time  $t$ . A complete set of dimensionless variables is

$$\omega_o t \quad \text{and} \quad \frac{b}{M\omega_o} \quad \text{with} \quad \omega_o = \sqrt{\frac{k}{M}} \quad (10)$$

So the

$$x(t) = x_o F\left(\omega_o t, \frac{b}{M\omega_o}\right) \quad (11)$$

where  $F$  is some function to be determined either numerically or analytically. As remarked in class (not by me) there are other parameters which are dimensionless. However these dimensionless parameters can be written as a combination of the complete set dimensionless parameters given above. For example

$$\frac{k}{b} t = \frac{k/M}{b/M} t = \frac{\sqrt{k/M}}{b/M} \sqrt{k/M} t = \frac{M\omega_o}{b} \omega_o t \quad (12)$$

Thus any function of  $(k/b)t$  is also a function of the dimensionless variables given above in Eq. 10.

Physically  $b/M\omega_o$  is the ratio between the damping rate  $b/M$ , and the oscillation frequency,  $\omega_o = \sqrt{k/M}$ . If the damping is small ( $b$  small), the damping rate is small we then

$$\frac{b}{M\omega_o} \ll 1. \quad (13)$$

This (dimensionless!) criterion is what we really mean by saying the damping is small.

## 3 Dimensionless variables

There is a another way to arrive at Eq. 11 which is useful for numerical work. We may choose to measure mass in units of  $M$ ; We can measure distances in  $x_o$ , and measure seconds in units of  $1/\omega_o$ . All other units can be derived from these quantities.

We will put a bar to denote a variable in this particular set of units. For example, since  $b$  has units [kg/s]. In our system of units we measure [kg/s] in  $M\omega_o$ . So we define  $\bar{b}$  as  $b$  divided by  $M\omega_o$  (*i.e.*  $\bar{b}$  is  $b$  in units of  $M\omega_o$ )

$$\bar{b} = \frac{b}{M\omega_o}. \quad (14)$$

The mass in units of  $M$  is simply,  $\bar{M} = M/M = 1$ . More generally

$$\bar{M} = 1 \quad \bar{k} = 1 \quad \bar{x}_o = 1 \quad \bar{t} = \omega_o t \quad \bar{b} = \frac{b}{M\omega_o}, \quad \bar{x} = \frac{x}{x_o}. \quad (15)$$

Other quantities can be expressed in these units. For instance the total energy has units of [kg m<sup>2</sup>/s<sup>2</sup>], which in our units would be expressed with  $Mx_o^2\omega_o^2 = kx_o^2$ . Thus the energy in our system of units is

$$\bar{E} = \frac{E}{kx_o^2}. \quad (16)$$

The velocity is

$$\bar{v} = \frac{v}{\omega_o x_o}. \quad (17)$$

Since the equation of motion is independent of what units you use we must have

$$\bar{M} \frac{d^2 \bar{x}}{d\bar{t}^2} + \bar{b} \frac{d\bar{x}}{d\bar{t}} + \bar{k} \bar{x} = 0, \quad \text{with} \quad \bar{x}(\bar{t} = 0) = \bar{x}_o. \quad (18)$$

Or substituting  $\bar{M} = \bar{k} = \bar{x}_o = 1$

$$\frac{d^2 \bar{x}}{d\bar{t}^2} + \bar{b} \frac{d\bar{x}}{d\bar{t}} + \bar{x} = 0 \quad \text{with} \quad \bar{x}(\bar{t} = 0) = 1. \quad (19)$$

Now we can solve this equation for  $\bar{x}$

$$\bar{x}(\bar{t}) = F(\bar{t}, \bar{b}). \quad (20)$$

After unraveling the definitions we conclude

$$x = x_o F(\omega_o t, \frac{b}{M\omega_o}). \quad (21)$$

We will solve Eq. 19 on the computer to determine this unknown function.

If you don't believe this units "trick" you can simply take the original equation

$$M\ddot{x} + b\dot{x} + kx = 0, \quad (22)$$

which has dimension [kg m/s<sup>2</sup>], and divide by the  $Mx_o\omega_o^2$  which has units of [kgm/s<sup>2</sup>] in our system of units

$$\frac{1}{Mx_o\omega_o^2} [M\ddot{x} + b\dot{x} + kx] = 0. \quad (23)$$

Look at the first term in this expression

$$\frac{1}{Mx_o\omega_o^2} M \frac{d^2 x}{dt^2} = \frac{d^2 \bar{x}}{d\bar{t}^2},$$

which is the first term in in Eq. 18 and Eq. 19. The other terms follow similarly, and the result is Eq. 18 and Eq. 19.

## 4 Numerical Scheme

We will solve Eq. 19 numerically for  $\bar{x}$  for various values of  $\bar{b} = b/M\omega_o$ . The first step is to rewrite Eq. 18 as two first order equations. Defining  $\bar{v} = d\bar{x}/d\bar{t}$  we have

$$\begin{aligned}\frac{d\bar{x}}{d\bar{t}} &= \bar{v} \\ \frac{d\bar{v}}{d\bar{t}} &= \frac{\bar{F}}{\bar{M}}\end{aligned}$$

where

$$\bar{F} = -\bar{k}\bar{x} - \bar{b}v \quad (24)$$

Since  $\bar{M} = \bar{x}_o = \bar{k} = 1$ , this system of equation has one free parameter  $\bar{b}$ .

The basic rule is the following: Given the position at velocity  $(x_n, v_n)$  at a time  $t_n = n\Delta t$ , we can find the position and velocity  $(\bar{x}_{n+1}, \bar{v}_{n+1})$  at time  $\bar{t}_{n+1} = \bar{t}_n + \Delta t$ , with the following step

$$\begin{aligned}\bar{x}_{n+1} &= \bar{x}_n + \bar{v}_n \Delta t \\ \bar{v}_{n+1} &= \bar{v}_n + \left(\frac{\bar{F}}{\bar{M}}\right)_n \Delta t\end{aligned}$$

We should choose the time step to be smaller than all time scales. This means, we should take

$$\omega_o \Delta t \ll 1, \quad \text{and} \quad \frac{b}{M} \Delta t \ll 1. \quad (25)$$

In dimensionless variables we have

$$\Delta \bar{t} \ll 1, \quad \text{and} \quad \bar{b} \Delta \bar{t} \ll 1. \quad (26)$$

A `matlab` program to use this update rule to solve for  $\bar{x}$  and  $\bar{v}$  as a function of  $\bar{t}$  is included at the end of this note and you should study it and understand how it works.

### 4.1 Results for zero damping

First we will take  $\bar{b} = 0$ . Then using the `matlab` program, we integrate from  $\bar{t} = 0 \dots 10$  with a time step of  $\omega_o \Delta t = 0.003$ . Figure 1 shows the position versus time. For comparison we also show the analytic answer which is simply

$$\frac{x}{x_o} = \cos(\omega_o t).$$

To test how well energy is conserved by our numerical procedure we show, the kinetic, potential, and total energies as a function of time in the appropriate unites. Figure 2 shows how well energy is conserved Energy appears nicely conserved in this approximation scheme.

To estimate the period we ask the program to print out the time whenever the velocity crosses zero. Note that when the velocity crosses zero  $\bar{x}(\bar{t})$  reaches a maximum or minimum. The program works by storing the last velocity (`vnold`) and the current velocity (`vn`) and comparing the sign of these two values with a simple set of statements

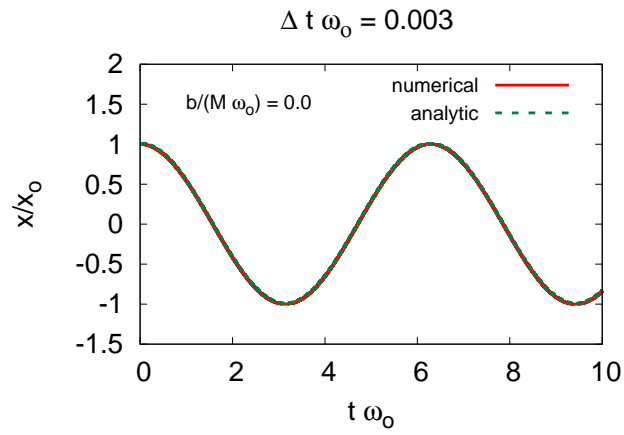


Figure 1: The position as a function of time for a harmonic oscillator without damping. The dashed curve shows the analytic result.

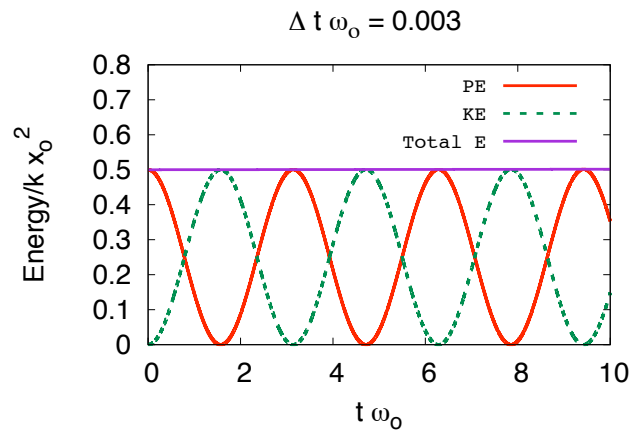


Figure 2: The kinetic energy, potential, and total energies as a function of time

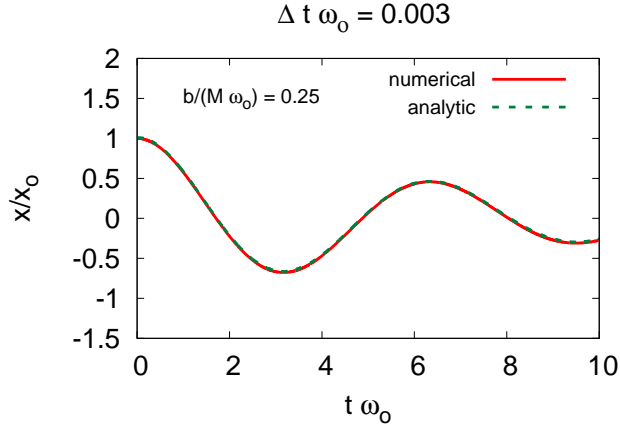


Figure 3: Damped harmonic oscillator for modest damping  $b/M\omega_o = 0.25$ . The numerical result is compared to the analytic result for of Eq. 8.

```

if vn * vnold < 0.
    % print out the result for the time
    vn
end

```

From running the program, we get a period of

$$\tau_{\text{osc}} = 6.285 \omega_o^{-1} \quad (27)$$

which should be compared with the exact answer

$$\tau_{\text{osc}} = \frac{2\pi}{\omega_o} \approx 6.283185 \omega_o^{-1} \quad (28)$$

Considering that our step size was 0.003 we should expect errors in the oscillation period of this order.

## 4.2 Results for non-zero damping

In the introduction we alluded to the fact that depending on the strength of the damping term we get different behaviors. Here we will illustrate this with our numerical work.

For modest damping, say

$$\bar{b} = \frac{b}{M\omega_o} = 0.25 \quad (29)$$

we get typically oscillatory behavior. The numerical solution to the differential equation in this case is shown in Figure 3

For  $b/M\omega_o \gg 1$ , the damping is large and we relax very slowly to equilibrium. Figure 4 shows highly overdamped motion for the rather large damping

$$\bar{b} = \frac{b}{M\omega_o} = 8.0 \quad (30)$$

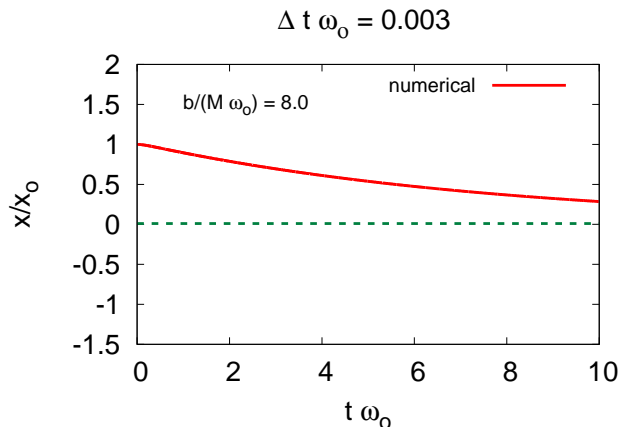


Figure 4: Damped harmonic oscillator for highly overdamped motion,  $b/M\omega_o = 8.0$ .

In this case the spring does not oscillate but relaxes slowly. The solution is not described by Eq. 8.

Finally, we turn to the critical value for the damping coefficient  $b/M\omega_o = 2$ . In this case the block comes down relatively quickly but does not cross the equilibrium point as is shown in Figure 5

### A Solving the equation analytically<sup>1</sup>

In this appendix we will show how the general solution to the equation of motion Eq. 1 can be obtained.

We are motivated by Figure 3 to search for a solution of the following form

$$x(t) = A(t) \cos(\omega t - \phi). \quad (31)$$

Substituting this into the equation of motion (and setting  $\phi = 0$  for simplicity) we get three terms

$$m \overbrace{\left[ \ddot{A} \cos(\omega t) - 2\omega \dot{A} \sin(\omega t) - \omega^2 A \cos(\omega t) \right]}^{\ddot{x}} + b \overbrace{\left[ \dot{A} \cos(\omega t) - \omega A \sin(\omega t) \right]}^{\dot{x}} + k \overbrace{\left[ A \cos(\omega t) \right]}^{x(t)} = 0.$$

Collecting the sin terms we get

$$-2m\dot{A}(t) - bA(t) = 0, \quad (32)$$

or

$$\frac{dA}{dt} = -\frac{b}{2m}A(t). \quad (33)$$

---

<sup>1</sup>This section can be skipped if the reader is pressed for time.

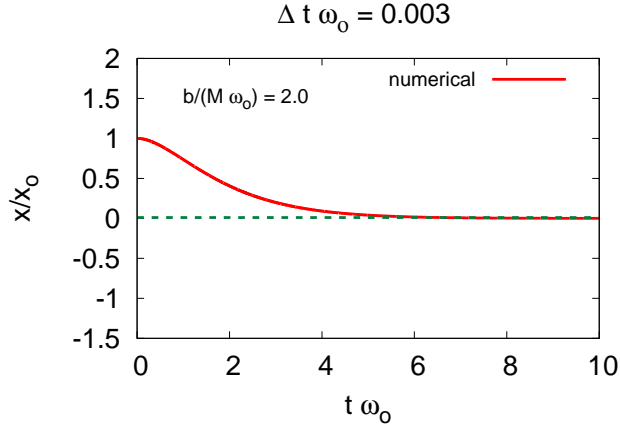


Figure 5: The case of critically damped motion,  $b/M\omega_o = 2$ .

A solution to this decay rate equation is

$$A(t) = A_o e^{-\frac{b}{2m}t}. \quad (34)$$

With this result we have

$$\dot{A} = -\frac{b}{2M}A(t) \quad \text{and} \quad \ddot{A} = \left(\frac{b}{2m}\right)^2 A(t). \quad (35)$$

Using these relation we collect the cos terms

$$\left\{ m \left[ \frac{b^2}{4m^2} - \omega^2 \right] + b \left[ -\frac{b}{2m} \right] + k \right\} A \cos(\omega t) = 0. \quad (36)$$

Solving for the frequency we get

$$\omega = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}. \quad (37)$$