Review of Complex Numbers

1. Show where in the complex plane are 1,i, $-1+\sqrt{3}i$, $\sqrt{i}$, $\sqrt{\sqrt{i}}$ and their complex conjugates. Compute the modulus $|z|^2$ of each case

2. From $e^{i(a+b)} = e^{ia}e^{ib}$, deduce the familar (song-based) rules for $\sin(a+b)$ and $\cos(a+b)$.

3. Also show that $e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos((a-b)/2)$ and deduce the somewhat less familiar

$$\cos(a) + \cos(b) = 2\cos((a+b)/2)\cos((a-b)/2) \tag{1}$$

$$\sin(a) + \sin(b) = 2\sin((a+b)/2)\cos((a-b)/2)) \tag{2}$$

   Discuss the physical significance of this result

4. Show that

$$\frac{1}{x+iy} = \underbrace{\frac{x}{x^2+y^2}}_{a} + i\,\underbrace{\frac{-y}{x^2+y^2}}_{b}$$

5. (a) Show $1+i = \sqrt{2}e^{i\pi/4}$ and $1-i = \sqrt{2}e^{-i\pi/4}$ (b) Show $|e^{ikx}|^2 = 1$ (c) Show $|e^{ik_1x}+e^{ik_2x}|^2 = 2(1+\cos(\Delta k\, x))$ with $\Delta k = k_1 - k_2$ (d) A general wave funtion is $\Psi(x) = R(x) + iI(x)$ where $R(x)$ and $I(x)$ are real functions. Show that $|\Psi|^2$ is positive. (e) A general wave function is $\Psi(x) = A(x)e^{i\phi(x)}$ where $A(x)$ and $\phi(x)$ are real functions, show that $|\Psi(x)|^2 = A(x)^2 = R(x)^2 + I(x)^2$.

6. **Ultra-Important:** Compute the "n-th" derivative of e^{ikx}. Start with one derivative and then generalize

$$(-i\frac{d}{dx})^n e^{ikx} \tag{3}$$

Extra problems not on complex numbers quiz

1. Show that

$$F(k) = \int_{-\infty}^{\infty} dx\, e^{ikx}e^{-a|x|} = \frac{2a}{k^2+a^2} \tag{4}$$

   Hint compute the integral from $-\infty$ to zero and zero to infinity. You will need to rationalize the denominators as in problem three. Integrals of the form

$$F(k) = \int_{-\infty}^{\infty} dx\, e^{ikx} f(x) \tag{5}$$

   are knowns as fourier integrals and are very important in all fields of science.

2. Consider the function of time and position

$$\psi(x,t) = e^{-i\omega t}F(x)\,. \tag{6}$$

   For definiteness take $F(x) = \sin(kx)$, though any *real* function of x will do. Qualitatively describe the imaginary part of this function, i.e. what does it do as function of time. Qualitatively, why does this function describe a standing wave. Now consider

$$\psi(t,x) = e^{-i\omega t + ikx}\,. \tag{7}$$

   Qualitatively describe this function as a function of time. why does this function describe a travelling wave

Complex Numbers

If complex numbers are completely foreign to you, you must consult a more complete discussionin any precalc or calc book.

1. A complex number

$$z = x + iy = re^{i\theta} = r\cos\theta + i\sin\theta$$

This is represented in the complex plane as shown below.

2. When one multiplies complex numbers the moduli (i.e. r) multiply and the angles add.

$$z_1 z_2 = r_1 e^{i\theta_1} r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$$

So multiplying by a pure phase $e^{i\phi}$ rotates the vector z by the angle ϕ

3. The complex conjugate of a complex number changes the sign of i

$$z^* = x - iy = re^{-i\theta} = r\cos\theta - i\sin\theta$$

and we note that $(z_1 z_2)^* = z_1^* z_2^*$

4. We used the important identity

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-i\theta} = \cos\theta - i\sin\theta$$

and the inverse relations

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

5. The modulus of complex number

$$|z|^2 \equiv z^* z = (x - iy)(x + iy) = x^2 + y^2 = r^2$$

We make the following notes about Modulus

   (a) The modulus of a pure phase is one $|e^{i\theta}| = 1$ In quantum mechancics the fact that the modulus of a pure wave (i.e. a single momentum) is one

   $$|e^{ikx}|^2 = 1$$

   says that the electron is equally likely to be anywhere, i.e. $\Delta k = 0$ and $\Delta x = \infty$.

   (b) The modulus of a product is the product of the moduli

   $$|ab| = |a||b|$$