Quiz

① Write

$z = \frac{1}{\sqrt{3} + i}$ in polar form, and draw z and

z^* in the complex plane.

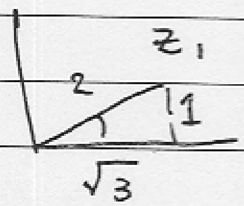
② Simplify $|e^{ik_1x} + i e^{ik_2x}|^2$ and graph
as fcn of x

③ Write $\sin(a) + \sin(b)$ as a product of sin's + cos'

Solution

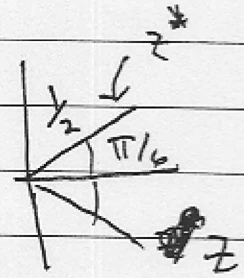
①

$$z = \frac{1}{\sqrt{3}+i}$$



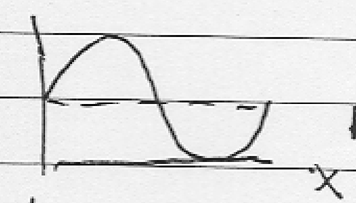
$$z_1 \equiv \sqrt{3} + i = 2\, e^{i\pi/6}$$

$$z = \frac{1}{z_1} = \frac{1}{2}\, e^{-i\pi/6}$$



$$z^* = \frac{1}{2}\, e^{i\pi/6}$$

② $$|e^{ik_1x} + ie^{ik_2x}|^2 = (e^{-ik_1x} - ie^{-ik_2x})(e^{ik_1x} + ie^{ik_2x})$$



$$= 1 + 1 - ie^{i(k_1-k_2)x} + ie^{i(k_2-k_1)x}$$

$$= 2 - i\left[e^{i(k_1-k_2)x} - e^{i(k_2-k_1)x}\right]$$

$$= 2 - i\, 2i\, \sin((k_1-k_2)x)$$

$$= 2 + 2\sin((k_1-k_2)x)$$

③ $$\sin(a) + \sin(b) = \mathrm{Im}\left[e^{ia} + e^{ib}\right]$$

$$e^{ia} + e^{ib} = 2e^{i\frac{a+b}{2}} \cos\left(\frac{a-b}{2}\right)$$

$$\mathrm{Im}\left[e^{ia} + e^{ib}\right] = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$

Complex Quiz

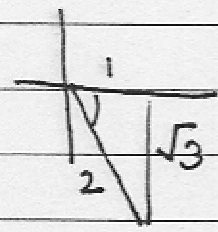
(1) Write $\dfrac{1}{1+\sqrt{3}i}$ in polar form, explicitly.

(2) Simplify $|e^{ik_1x} + e^{-ik_2x}|^2$

(3) Write $\cos(a) + \cos(b)$ as a product of sin's & co[s], derive the result.

(1) $$\frac{1}{1+\sqrt{3}i} = \frac{1}{(1+\sqrt{3}i)} \cdot \frac{1-\sqrt{3}i}{1-\sqrt{3}i} = \frac{1-\sqrt{3}i}{1^2+3}$$

$$= \frac{1}{4} - \frac{\sqrt{3}}{4}i = \frac{1}{2}e^{-i\pi/3}$$



$$r = \frac{1}{2} = \sqrt{\left(\frac{1}{4}\right)^2 + \left(\frac{\sqrt{3}}{4}\right)^2}$$

$$\varphi = \tan^{-1}\left(\frac{-\sqrt{3}/4}{1/4}\right) = -60^\circ$$

(2) $$\underbrace{(e^{-ik_1x} + e^{ik_2x})}_{=\psi^*}\underbrace{(e^{ik_1x} + e^{-ik_2x})}_{\psi} = 2 + e^{i(k_1+k_2)x} + e^{-i(k_1+k_2)x}$$

$$= 2 + 2\cos((k_1+k_2)x)$$

(3) $$e^{ia} + e^{ib} = e^{i(a+b)/2} e^{i(a-b)/2}$$

$$+ e^{i(a+b)/2} e^{-i(a-b)/2}$$

$$= e^{i(a+b)/2} \left[e^{i(a-b)/2} + e^{-i(a-b)/2} \right]$$

So $$e^{ia} + e^{ib} = e^{i(a+b)/2}\, 2\cos\left(\frac{a-b}{2}\right)$$

Taking the real part:

$$\mathrm{Re}\left[e^{ia} + e^{ib}\right] = \mathrm{Re}\left[e^{i(a+b)/2}\right] 2\cos\left(\frac{a-b}{2}\right)$$

$$\boxed{\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)}$$

Alternate Proof:

$$\cos(a) + \cos(b) = \cos\Bigg(\overbrace{\frac{a+b}{2}}^{x} + \overbrace{\frac{a-b}{2}}^{y}\Bigg) + \cos\Bigg(\overbrace{\frac{a+b}{2}}^{x} - \overbrace{\frac{(a-b)}{2}}^{y}\Bigg)$$

$$\cos(x+y) = \cos x \cos y - \sin(x)\sin(y)$$

this changes sign and cancels in sum

$$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$$