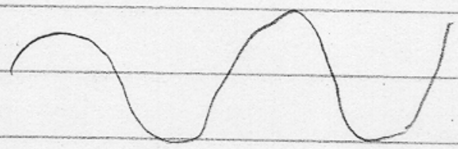
Last Time

① Electron is a wave -- Debroglie

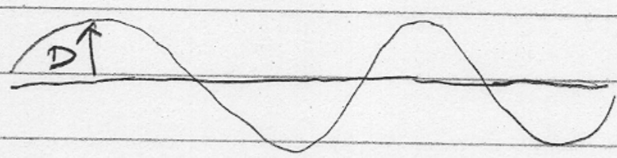


$$\lambda = \frac{h}{p}$$

For an electron with 10eV of $K = \frac{p^2}{2m}$
the wavelength is of order

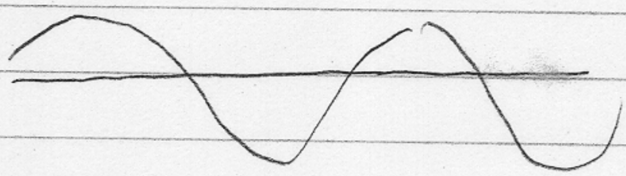
$$\lambda \sim 4\,\mathring{A}$$

② Analogy with waves on a string



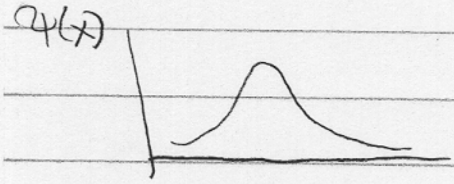
$$D(x,t) = A\cos\left(\frac{2\pi x}{\lambda} - 2\pi f t\right.$$

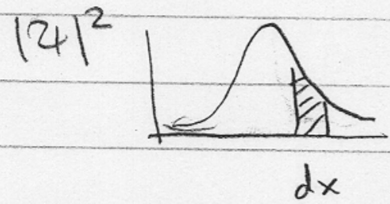
$\uparrow$
displacement



$\psi(x,t) \leftarrow$ electrons wave function

The physical meaning of $\psi(x)$ was initially unclear

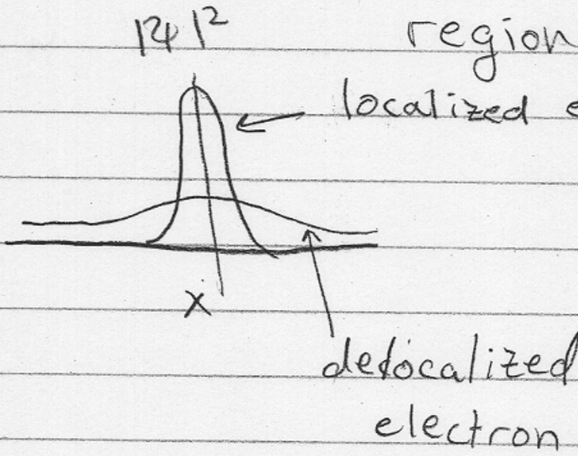




Today

$$dP = |\psi(x)|^2 dx$$ ← probability to find the electron between x and $x + dx$

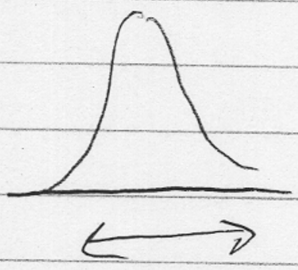
- Thus if an electron is localized in a small region around the origin, its wave-fcn is sharply peaked



If the electron is delocalized its wave function is spread out

- Much more on this later

(4) Uncertainty Principle



- If the electron wavelength is short (it is localized in some region), The momentum is large

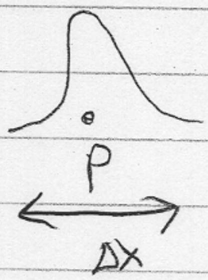
$$\lambda \sim \Delta x \qquad p \sim \frac{h}{\Delta x}$$

Usually written

$$\Delta p \, \Delta x \sim \hbar$$

(5) Bound States:

- Electron is attracted to the proton, so the attractive forces tend to localize the electron around the proton



- As the electron becomes localized, its momentum increases

$$p \sim \frac{\hbar}{\Delta x}$$

(5) Continued

- The kinetic energy increases and tries to blow the electron away from proton

$$K = \frac{1}{2} m v^2 = \frac{p^2}{2m} \sim \frac{\hbar^2}{2m\Delta x^2}$$

The potential energy is balanced by the kinetic energy

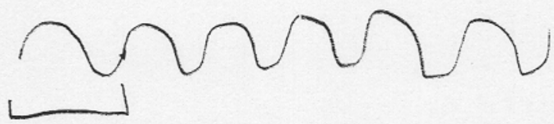
$$|PE| \simeq \frac{e^2}{4\pi\varepsilon_0 \Delta x} \sim KE \sim \frac{\hbar^2}{2m\Delta x^2}$$

So comparing:

$$\Delta x \sim \frac{\hbar^2}{2m\left(\frac{e^2}{4\pi\varepsilon_0}\right)}$$ ← This is of order the Bohr radius

$$a_0 = \frac{\hbar^2}{m(e^2/4\pi\varepsilon_0)}$$

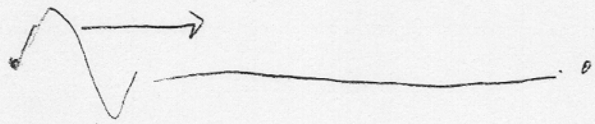
Debroglée



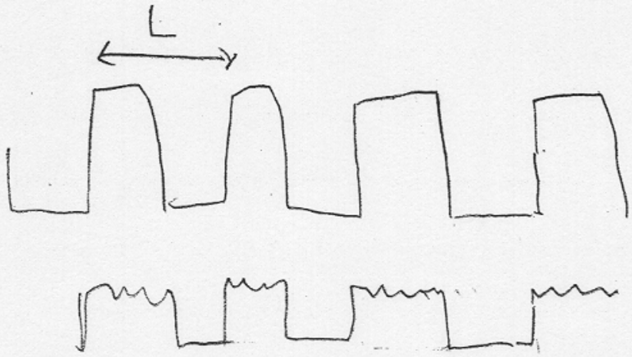
$$\lambda = \frac{h}{p}$$

What about a wave packet?

String



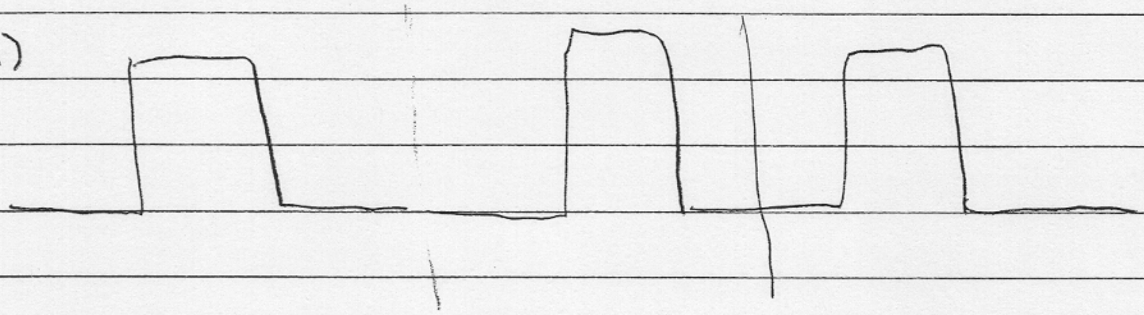
More about waves:



Claim: Any such waves can be written as a sum of sin/cos waves

Example:

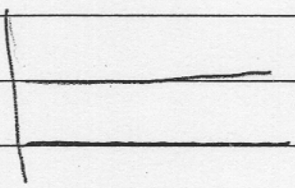
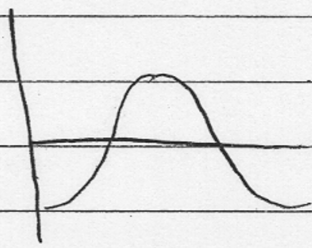
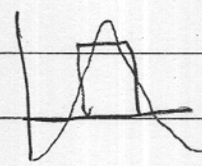
f(x)



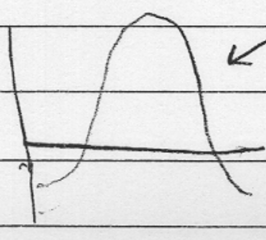
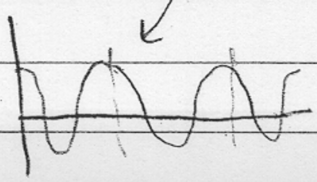
Focus on one window (See Handout)

① $f(x) \simeq 1 +$ corrections $=$ Fig 1

② $f(x) \simeq 1 + \frac{2}{\pi}\cos(2\pi x) =$ Fig 2

 +  = Fig 2 = 

③ $f(x) \simeq 1 + \underbrace{\frac{2}{\pi}\cos(2\pi x)} - \underbrace{\frac{2}{3\pi}\cos(2\pi \cdot 3x)} =$ Fig 3

 +  = Fig 3

④ many terms = Fig 4

Basic Observation

(1) - In order to reconstruct a waveform wich is small say of size Δx

- Need to superimpose many wavelength's

$$\lambda \sim \Delta x$$

- The waves are written

$$\cos(kx)$$

So $$k = \frac{2\pi}{\lambda} \sim \frac{2\pi}{\Delta x}$$

$$k\,\Delta x \sim 2\pi$$

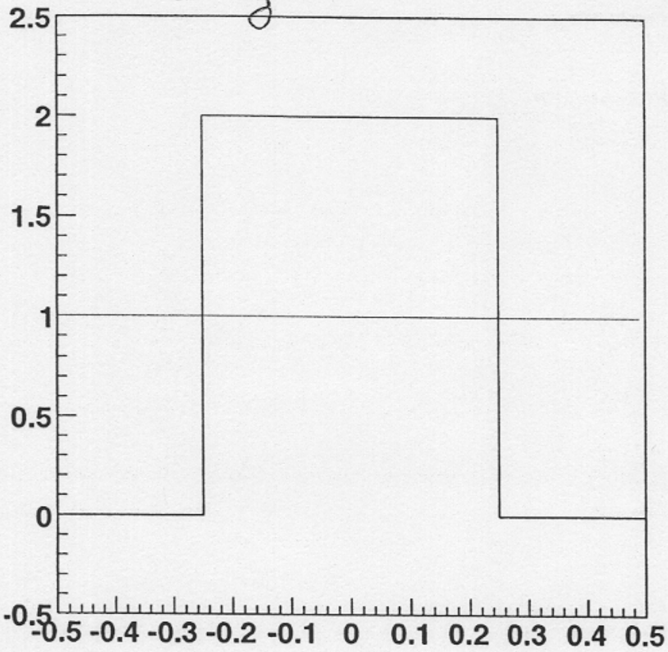
- Now multiply by = $\hbar$

$$\underbrace{\hbar k}_{p}\,\Delta x \sim 2\pi\hbar$$

$$p\,\Delta x \sim 2\pi\hbar$$

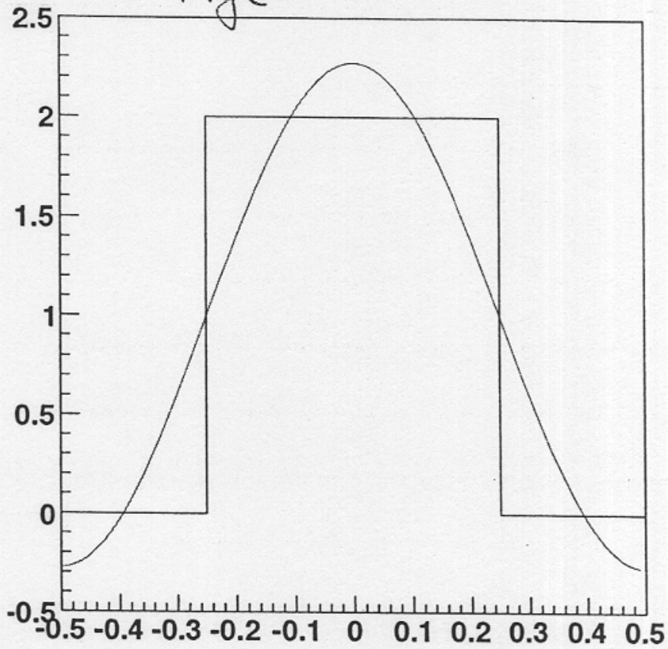
(2) There isn't one wavelength associated with a wave. There is a superposition of many wavelengths

Fig 1



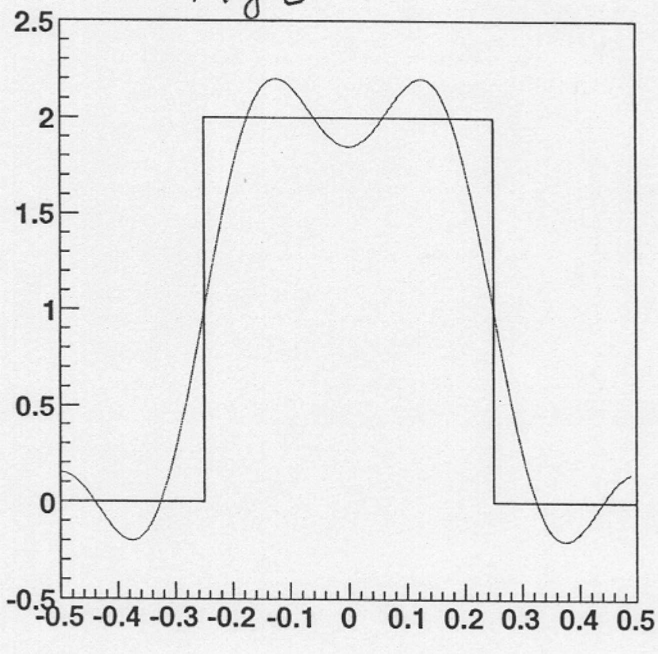
$f(x) \simeq 1 + \ldots\ldots$

Fig 2



$f(x) \approx 1 + \frac{2}{\pi}\cos(2\pi x) + , \ldots$

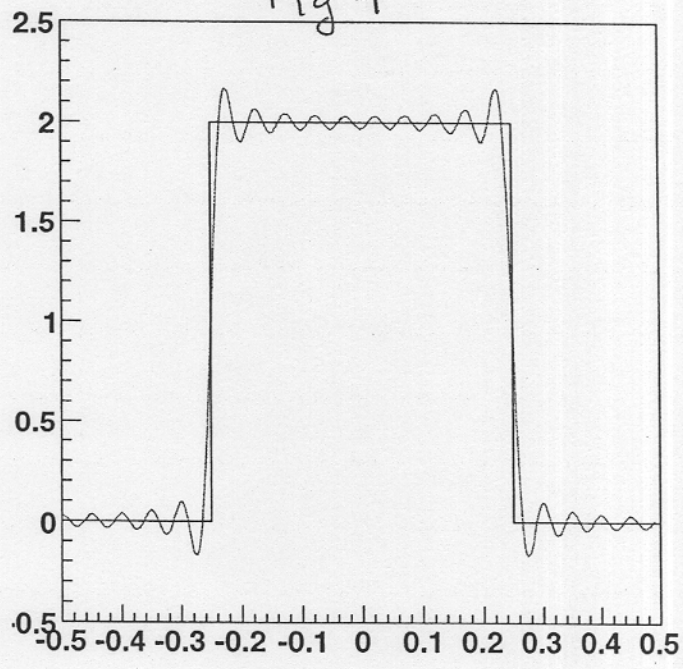
Fig 3



$f(x) \simeq 1 + \frac{2}{\pi}\cos(2\pi x) - \frac{2}{3\pi}\cos(2\pi x \cdot 3)$

$+ \ldots$

Fig 4



$f(x) = 1 + \frac{2}{\pi}\cos(2\pi x) - \frac{2}{3\pi}\cos(2\pi x\cdot$

$+ \frac{2}{5\pi}\cos(2\pi x) - \frac{2}{7\pi}\cos(2\pi x\cdot$

$+ \ldots$ many more

Simple Superposition of two waves (See handout)

$$\psi_1 = \sin(kx) = \sin\left(\bar{k}x - \frac{\Delta k}{2} x\right) \Rightarrow \sin(15x)$$

$$\psi_2 = \sin((k+\Delta k)x) = \sin\left(\bar{k}x + \frac{\Delta k}{2} x\right) \Rightarrow \sin(16x)$$

$$\bar{k} = \frac{k + (k+\Delta k)}{2} = k + \frac{\Delta k}{2} \qquad \bar{k} = 15.5$$

average

$$\sin(a-b) = \sin a \cos(+b) - \cos(a)\sin(+b)$$

$$+ \quad \sin(a+b) = \sin a \cos(b) + \cos(a)\cos(b)$$

$$\sin(a-b) + \sin(a+b) = 2\sin a \cos b$$

So

$\bar{k} \doteq 15.5$ $\quad \Delta k = 1$

$$\psi_1 + \psi_2 = \underbrace{2\sin(\bar{k}x)}_{\text{changes very rapidly}} \underbrace{\cos\left(\frac{\Delta k}{2} x\right)}_{\text{changes very slowly}}$$

changes very rapidly

$$\lambda_{fast} \simeq \frac{2\pi}{\bar{k}}$$

changes very slowly

$$\underbrace{\lambda_{slow}}_{2\Delta X} = \frac{2\pi}{\Delta k/2}$$

$\cancel{2}\, \Delta k\, \Delta x = 2\cancel{4}\pi$

$\Delta k\, \Delta x = 2\pi \xrightarrow{\text{multiply by } \hbar \;(p=\hbar k)} \boxed{\Delta p\, \Delta x = 2\pi\hbar}$ ← special case considered here

In general the spread in momentum is related to the spread in position

$$\boxed{\Delta p\, \Delta x \geq \frac{\hbar}{2}}$$

always true

Also applies to temporal waves

$\Psi_1 = \cos(\omega_1 t) + \cos(\omega_2 t)$

$\Psi_1 = \underbrace{2\sin\bar{\omega} t}_{\text{rapid variation}}\ \underbrace{\cos\left(\frac{\Delta\omega}{2} t\right)}$

beat frequency slow

Now as before:

$T_{slow} \sim \dfrac{2\pi}{\Delta\omega/2}$

$2\Delta t \sim \dfrac{4\pi}{\Delta\omega}$

$\Delta t\, \Delta\omega \sim 2\pi$

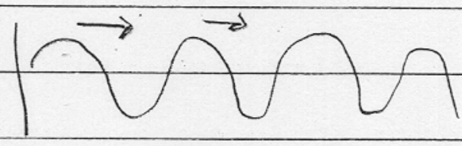
$\Delta t\, (\hbar\Delta\omega) \sim 2\pi\hbar$

$\Delta t\, \Delta E \sim \hbar$

$$\boxed{\Delta t\, \Delta E \geq \frac{\hbar}{2}}$$

Now consider moving waves

$\Psi = \sin(k_1 x - \omega_1 t)$



$k = \frac{2\pi}{\lambda} \qquad \omega = \frac{2\pi}{\tau} = 2\pi f$

The phase velocity

$$v = \lambda f = \frac{\lambda}{2\pi}(2\pi f) = \frac{\omega}{k} \equiv v_{phase}$$

Now Rember

$$E = \hbar\omega \qquad p = \hbar k$$

So there seems to be a problem

$$E = \frac{p^2}{2m}$$ (for a free electron no potential)

Then $v_{phase} = \frac{\hbar\omega}{\hbar k} = \frac{p^2/2m}{p} = \frac{p}{2m}$

Hmm? would have wanted $v = \frac{p}{m}$ not $\frac{p}{2m}$

How will the classical limit be achieved

Superposition of Moving Waves

$$\psi_1 = \sin(k_1 x - \omega_1 t) + \sin(k_2 x - \omega_2 t)$$

$$\sin(\theta_1) + \sin(\theta_2) = 2\sin\underbrace{\left(\frac{\theta_1+\theta_2}{2}\right)}_{\text{average}} \; + \; \cos\underbrace{\left(\frac{\Delta\theta}{2}\right)}_{\text{diff}}$$

So as Before:

$$\psi = \underbrace{2\sin(\bar{k}x - \bar{\omega}t)}_{\text{Rapid Oscillations}} \underbrace{\cos(\Delta k\, x - \Delta\omega t)}_{\text{envelope}}$$

- The Rapid Oscillatiations "motion inside the envelople"
- Envelope Itself moves with a speed

$$V_{group} = \frac{\Delta\omega}{\Delta k} = \frac{d\omega_1}{dk_1} = \frac{d(\hbar\omega_1)}{d(\hbar k)_1} = \frac{d(P_1^2/2m)}{dp_1} = \frac{p_1}{m} \simeq \frac{\bar{p}}{m}$$

- Then the envelope itself moves at the 'group' velocity

Summary

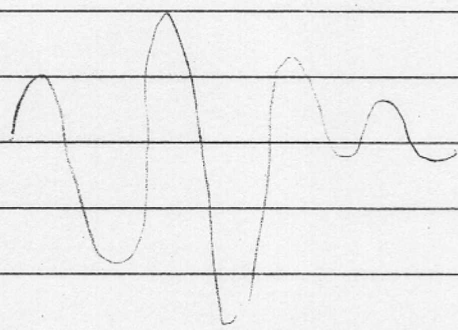
- free Electrons move as wave packets with

$$v = \frac{\bar{p}}{m}$$

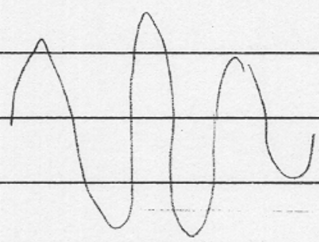
- The momentum we speak about when we say the particle has momentum such and such is really the average momentum $\bar{p}$ and the difference is small

$$\Delta p \ll \bar{p}$$

Waves and Electrons



So far we've been drawing pictures of waves:

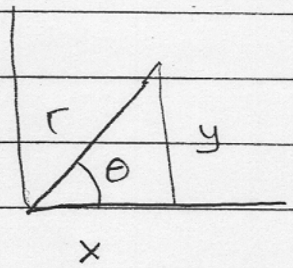


We mean there is a function $\psi(x,t)$ which tells something about the particle

Complex Numbers: Aside

$$z = x + iy = r\cos\theta + ir\sin\theta = r(\cos\theta + i\sin\theta)$$

$$= re^{i\theta} \qquad \theta = \tan^{-1} y/x$$



$$z^* = x - iy = r(\cos\theta - i\sin\theta) \qquad |z|^2 \equiv z^* z = (x+iy)(x-iy)$$

$$= x^2 + y^2 = r^2$$

changes the sign of i

Important

$$e^{i\theta} = \cos\theta + i\sin\theta \qquad e^{-}$$

Proof:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots$$

$$e^{ix} = 1 + ix = \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} + \ldots$$

Then

$$e^{ix} = \underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots\right)}_{\cos x} + i \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots\right)}_{\sin x}$$

$$e^{ix} = \cos x + i \sin x$$

The length of a pure phase is one

$$|e^{+i\theta}|^2 = e^{-i\theta} e^{+i\theta} = e^0 = 1$$

$$= (\cos\theta + i\sin\theta)^* (\cos\theta + i\sin\theta)$$

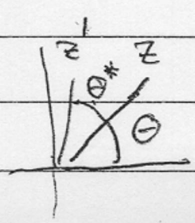
$$= (\cos\theta - i\sin\theta)(\cos\theta + i\sin\theta)$$

$$= \cos^2\theta + \sin^2\theta \cancel{- i\sin\theta\cos\theta + i\sin\theta\cos\theta}$$

$$= 1$$

Multiplication By a phase : $z = re^{i\theta}$

$$z' = z e^{i\theta_*} = re^{i\theta} e^{i\theta_*} = r e^{i(\theta + \theta_*)}$$



Anything Done (w) sin's and cos's is better done (w) $e^{i\theta}$

Example:

$\psi_1 = \cos(k_1 x)$ $\quad\quad$ $\psi_2 = \cos(k_2 x)$

$\psi_1 = \text{Re}\ e^{ik_1 x} = \text{Re}\left[\cos kx + i \sin kx\right] = \cos kx$

$\psi_2 = \text{Re}\ e^{ik_2 x}$

$\psi_1 + \psi_2 = \text{Re}\left[e^{ik_1 x} + e^{ik_2 x}\right]$

$k_1 = \bar{k} - \frac{\Delta k}{2}$ $\quad\quad$ $k_2 = \bar{k} + \frac{\Delta k}{2}$ $\quad\quad$ $\Delta k = k_1 - k_2$

$\psi_1 + \psi_2 = \text{Re}\left[e^{i(\bar{k} - \frac{\Delta k}{2})x} + e^{+i(\bar{k} + \Delta k/2)x}\right]$

$\psi_1 + \psi_2 = \text{Re}\left[e^{i\bar{k}x} \cdot \left(e^{-i\frac{\Delta k}{2}x} + e^{+i\frac{\Delta k}{2}x}\right)\right]$

$= \text{Re}\left[e^{i\bar{k}x} \cdot 2\cos \frac{\Delta k}{2} x\right]$

$= 2\cos \frac{\Delta k}{2} x \ \text{Re}\left[e^{i\bar{k} x}\right] = 2\cos \frac{\Delta k}{2} x \ \cos(\bar{k} x)$