Last Time

① Schrödinger Equation:

$$\left[\frac{\mathbb{P}^2}{2m} + V(x)\right]\Psi = \mathbb{E}\,\Psi$$

$$\underbrace{\quad}_{KE} \qquad \underbrace{\quad}_{PE} \qquad \underbrace{\quad}_{E}$$

$$\mathbb{P} = -i\hbar\frac{\partial}{\partial x} \qquad \mathbb{E} = +i\hbar\frac{\partial}{\partial t}$$

$$\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi = +i\hbar\frac{\partial \Psi}{\partial t}$$

Then looked for waves with one frequency (standing waves)

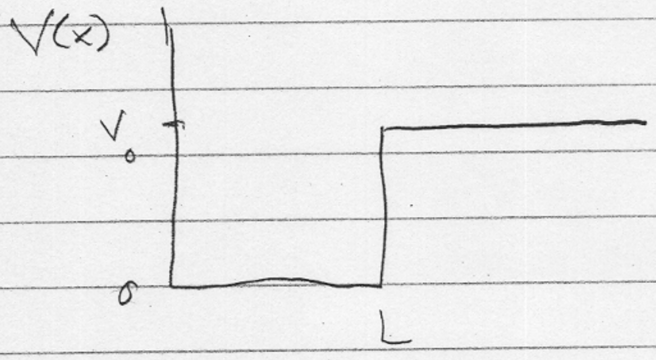
$$\Psi(x,t) = \Psi_n(x)\,e^{-iE_n t/\hbar}$$

Find the time indep schrödinger eq:

$$\underbrace{\left[-\frac{\hbar^2}{2m}\frac{\partial^2}{\partial x^2} + V(x)\right]\Psi_n(x) = E_n\,\Psi_n(x)}$$

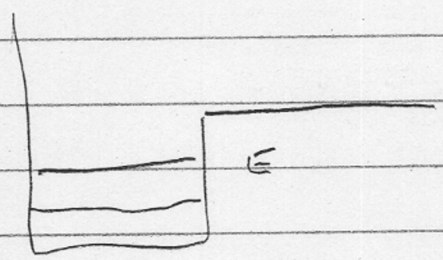
mostly what we will work with

(2) Then we started studying




- describes short range attraction, e.g the attraction of a proton to a nucleus

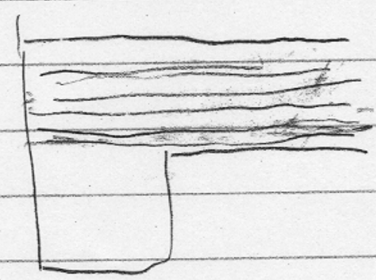
Classical vs. Quantum:




neutron bouncing inside a nucleus with periodic orbits

$E < V_0$

discrete Energies



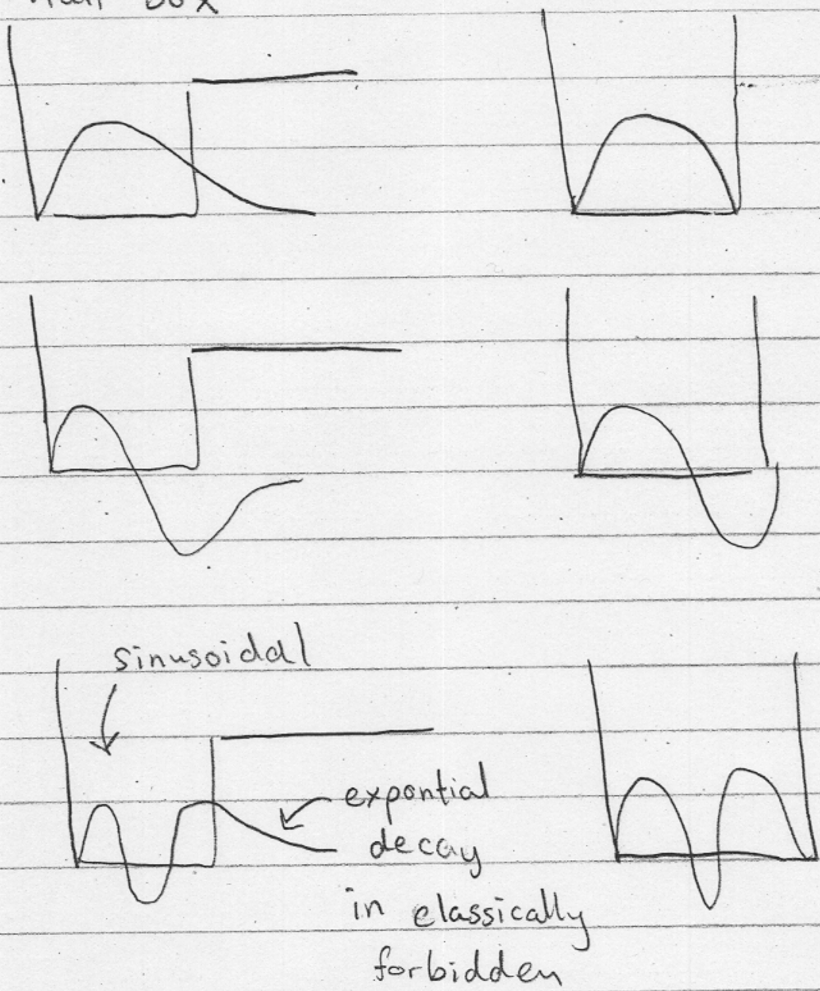
neutron scattering off nucleus

$E > V_0$

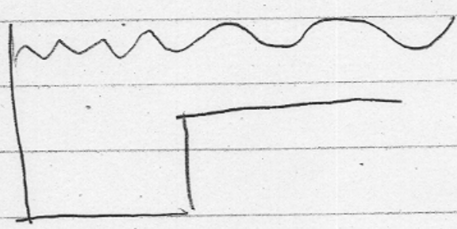
continuous energies

## ③ Qualitative – Bound State

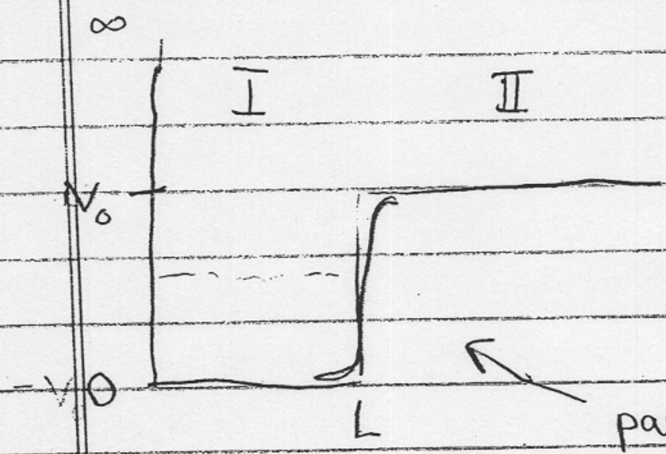
Half Box Box




For $E > V_0$



Now consider Particle in a half box




$$\frac{V_0}{\frac{\hbar^2}{2mL^2}} = 30$$

$$\frac{V_0}{\hbar/mL^2} = 15$$

particle in a swimming pool

or neutron in a nucleus!

Lets Solve the time indep. Schrödinger equation for this problem:

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V(x)\right]\Psi = E\Psi$$

Region I.

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + 0\right]\Psi = E\Psi$$

$$\frac{d^2\Psi}{dx^2} = \underbrace{\left(\frac{2mE}{\hbar^2}\right)}_{\equiv k^2}\Psi$$

$$\Psi = A\sin(kx) + B\cos(kx)$$

Now the wave fcn must vanish at the end of the box

$$\psi(x)\Big|_{0} = 0$$

So must have $B = 0$

$$\psi = A\sin(kx)$$

Lets set $A = 1$, Later we will normalize the wave fcn

$$\psi = \sin(kx) \quad \Leftarrow \text{un-normalized}$$

<u>Region II</u>

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V_0\right]\psi = E\psi$$

$E < V_0$ ← the particle is in the pool

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - V_0)\psi$$

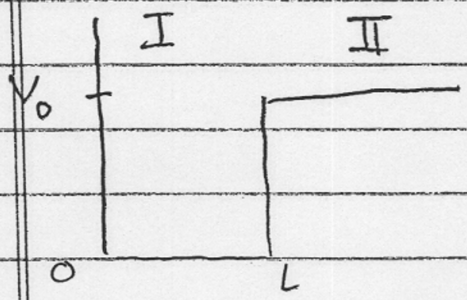
$$\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = +\underbrace{(V_0 - E)}_{\text{positive}}\psi$$

$$\frac{d^2\psi}{dx^2} = \underbrace{\frac{2m}{\hbar^2}(V_0 - E)}_{\equiv \bar{k}^2}\psi$$

Solution

$$\psi(x) = \bar{A}\, e^{\bar{K}x} + \bar{B}\, e^{-\bar{K}x}$$

Summary




$$\text{I}\quad \psi(x) = \begin{cases} \sin(kx) & k = \left(\dfrac{2mE}{\hbar^2}\right)^{1/2} \text{ in I} \\[2ex] \bar{A}e^{\bar{K}x} + \bar{B}e^{-\bar{K}x} & \bar{K} = \left(\dfrac{2m(V-E)}{\hbar^2}\right)^{1/2} \text{ in} \end{cases}$$

Now at $x = L$ must have $\qquad \bar{K} = \left(\dfrac{2mV_0}{\hbar^2} - k^2\right)^{1/2}$

$$\psi_{\text{I}}\Big|_L = \psi_{\text{II}}\Big|_L$$

$$\frac{d\psi_{\text{I}}'}{dx}\Big|_L = \frac{d\psi_{\text{II}}}{dx}\Big|_L$$

So

$$\sin(kL) = \bar{A} e^{\bar{k}L} + \bar{B} e^{-\bar{k}L}$$

$$k\cos(kL) = \bar{A}\bar{k} e^{\bar{k}L} - \bar{B}\bar{k} e^{-\bar{k}L}$$

Solve for $\bar{A}, \bar{B}$:

$$\bar{A} = \left[ \sin(kL) + \frac{k}{\bar{k}} \cos(kL) \right] \frac{e^{-\bar{k}L}}{2}$$

$$\bar{B} = \left[ -\frac{k}{\bar{k}} \cos(kL) + \sin(kL) \right] \frac{e^{+\bar{k}L}}{2}$$

- Now we want $A = 0$ so

$$-\sin(kL) = \frac{k}{\bar{k}} \cos(kL)$$

$$\sqrt{\frac{2mV_0}{\hbar^2} - k^2} = -k\cot(kL)$$

$$(*) \quad \sqrt{\left(\frac{2mL^2}{\hbar^2}\right)V_0 - (kL)^2} = -(kL)\cot(kL)$$

$$E = \frac{\hbar^2}{2mL^2}(kL)^2 = \frac{\hbar^2}{2mL}(x)^2 \qquad x \equiv kl$$

Now take for Example

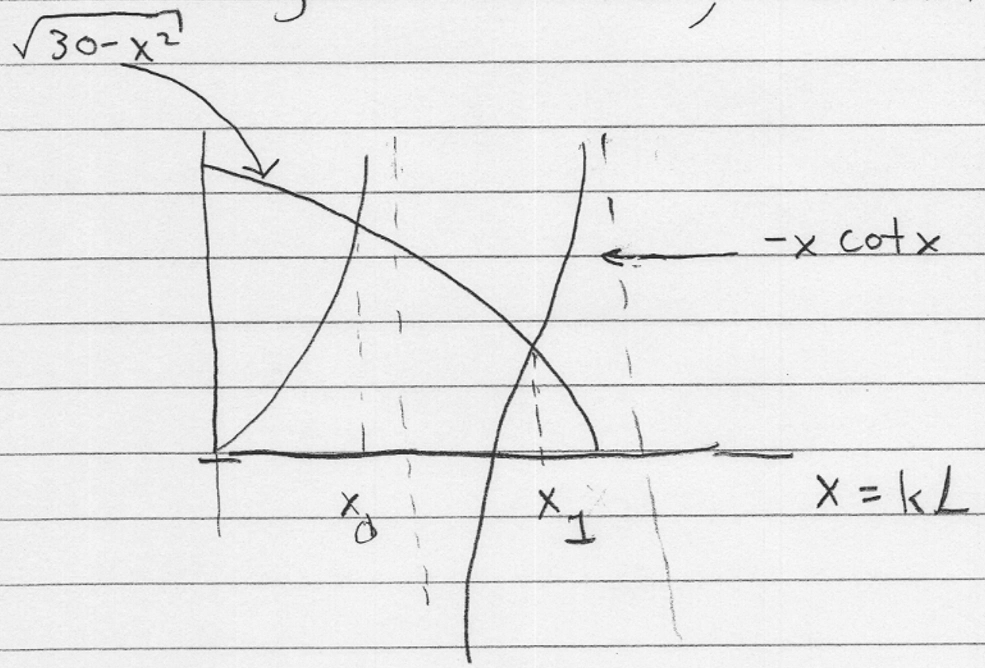
$$\frac{V_0}{\hbar/mL^2} = 15 \qquad \text{so} \qquad \frac{2mL^2}{\hbar^2} V_0 = 30$$

Then defining $x \equiv kL$,

Equation (*) on the previous page reads

$$\sqrt{30 - x^2} = -x \cot x$$ ← to be solved for $x$

No analytic Solution, but ... graphically




$x_0 = 2.63 \qquad x_1 = 5.09$ from graph

Then

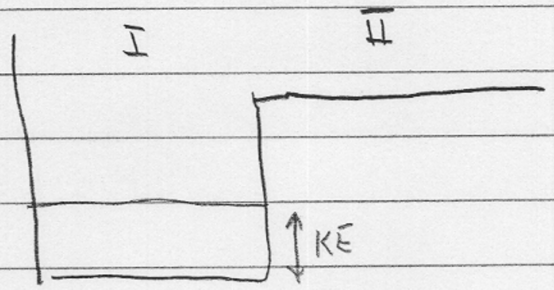
$$E = \frac{\hbar^2}{mL^2}\left(\frac{1}{2}x^2\right)$$

$$E_0 = \frac{\hbar}{mL^2}\left(\frac{1}{2}x_0^2\right) = \frac{\hbar^2}{mL^2}(3.4818) \quad \leftarrow \text{ground state energy}$$

$$E_1 = \frac{\hbar^2}{mL^2}\left(\frac{1}{2}x_1^2\right) = \frac{\hbar^2}{mL^2}(12.96) \quad \leftarrow \text{first excited state}$$

<u>Summary</u>:

- These features are not specific to this problem




<u>Classically allowed I</u>:

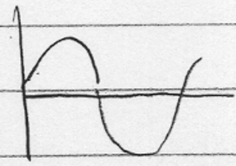
- Sinusoidal oscillations: wavelength controlled by free Kinetic Energy

$$\frac{\hbar^2 k^2}{2m} = E - V(x)$$

$$k = \sqrt{\frac{2m}{\hbar^2}(E - V(x))}$$

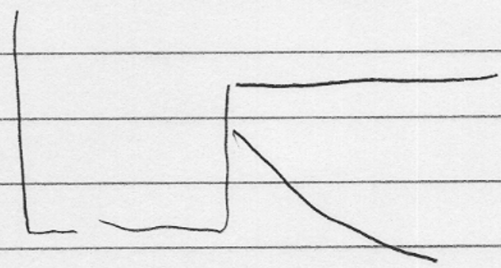
more generally:

$$\frac{d^2\psi}{dx^2} = -\frac{2m}{\hbar^2}(E - V)\psi$$



$$\frac{d^2\psi}{dx^2} = \text{concavity or curvature} = -\frac{2m(E-V)}{\hbar^2}\psi$$

Classically Forbidden II



i.e that it fit in box

- Require that the wave is decreasing, Leads to quantizat of Energy

$$\psi(x) \propto C e^{-\bar{k}(x-L)}$$

$$\bar{k} = \sqrt{\frac{2m}{\hbar^2}(V(x) - E)}$$

The "penetration depth" is how far the wave goes into the classical region

$$D \equiv \frac{1}{K} \sim \sqrt{\frac{\hbar^2}{2m(V-E)}}$$