Summary

$$(\text{mass of combo})c^2 = \left(\underset{\text{far apart}}{\text{mass of constituents}}\right)c^2 + \overbrace{BE}^{\text{Binding energy}}$$

$$(\text{mass of combo})c^2 < \left(\underset{\text{far apart}}{\text{mass of constituents}}\right)c^2 \quad \text{for bound states}$$

Energy and Momentum:

$E_0 = mc^2$ **Rest** ($E_0 = mc^2$) **Moving:** ($E = ?$ $p = ?$)

E_0 = Energy of a particle at rest

- What about kinetic energy & momentum when moving
- What is the velocity of a <u>slow</u> particle:

$$p = mu \implies u = \frac{p}{m} = \frac{c^2 p}{mc^2} = \boxed{c^2 \frac{p}{E} = u}$$

This also works for light:

$$u = c^2 \frac{p}{E} = c\left(\frac{cp}{E}\right) = c \quad \checkmark$$

So $\boxed{u = c^2 \frac{p}{E}}$

- Now suppose we do work on the particle to increase its energy

$$dE = F\,dx \qquad \text{now} \qquad F = \frac{dp}{dt}$$

$$dE = \frac{dp}{dt}dx = dp\,u$$

So

$$dE = c^2 \frac{p}{E}\,dp$$

or

$$E\,dE = c^2 p\,dp$$

integrate both sides

$$\frac{1}{2}E^2 = \frac{1}{2}c^2p^2 + \text{Constant}$$

the constant is different between these lines

$$E^2 = c^2p^2 + \text{Constant}$$

Now when the partice is at rest $p=0$, $E=E_0$

$$E_0^2 = c^2\cancel{p^2} + \text{Constant} = (m_0c^2)^2$$

So

$$E^2 = c^2p^2 + (m_0c^2)^2$$

$$\boxed{E = \sqrt{(cp)^2 + (m_0c^2)^2}}$$

For a photon $m=0$

$E = cp$

Then we can use

$$E^2 = (cp)^2 + (m_0 c^2)^2 \qquad \text{and} \qquad u = c^2 \frac{p}{E}$$

$$\text{or} \qquad \frac{u^2}{c^2} E^2 = (cp)^2$$

To multiply:

$$\underbrace{\frac{u^2}{c^2} E^2} = \frac{v^2}{c^2}(cp)^2 + \frac{v^2}{c^2}(mc^2)^2$$

$$(cp)^2 = \frac{u^2}{c^2}(cp)^2 + \frac{u^2}{c^2}(mc^2)^2$$

Solving for cp:

$$(cp)^2 = \frac{u^2/c^2 \, (m_0 c^2)^2}{1 - u^2/c^2}$$

$$cp = \frac{\frac{u}{c} \, m_0 c^2}{\sqrt{1 - u^2/c^2}}$$

$$\boxed{p = \gamma m_0 u} \qquad \gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

Similarly we have: $\qquad p = \gamma m v$

$$E^2 = c^2p^2 + (m_0c^2)^2$$

$$c^2p^2 = \gamma^2 (mc^2)^2 \cdot (u/c)^2$$

$$c^2p^2 = \frac{(m_0c^2)^2 (u/c)^2}{(1-(u/c)^2)}$$

$$E^2 = \frac{(mc^2)^2 (u/c)^2}{1-(u/c)^2} + (m_0c^2)^2$$

$$E^2 = \frac{(m_0c^2)^2}{1-(u/c)^2}$$

$$\boxed{E = \gamma m_0 c^2} \qquad \gamma \equiv \frac{1}{\sqrt{1-(u/c)^2}}$$

Example

Protons circle the LHC with a energy of $E = 5$ TeV. ~~Dertermine~~ the "gamma" factor.
Determine the velocity

$E = \gamma m c^2$ so

$$\gamma = \frac{E}{mc^2} = \frac{5 \times 1000 \text{ GeV}}{.939 \text{ GeV}}$$

1 Giga $= 10^9$

1 Tera $= 10^{12}$

$$\gamma \simeq 5000$$

To find the velocity:

$$\gamma = \frac{1}{\sqrt{1-(u/c)^2}} \Rightarrow \gamma^2 = \frac{1}{1-(u/c)^2}$$

These steps have been discussed so just go quickly

$$\frac{1}{\gamma^2} = 1-(u/c)^2$$

$$(u/c)^2 = 1 - \frac{1}{\gamma^2}$$

$$u/c = \sqrt{1-1/\gamma^2} \simeq 1 - \frac{1}{2}\frac{1}{\gamma^2}$$

$$u/c \simeq 1 - \frac{1}{2}\frac{1}{(5\times 10^3)^2}$$

$$u/c \simeq 1 - 2\times 10^{-8}$$

<u>Limit of Non-rel</u>

Should Find for $u/c \ll 1$ $\quad KE \simeq \frac{1}{2}mu^2$, $p = mu$

etc

Momentum:

$$p = \gamma m u \qquad \gamma \simeq \frac{1}{\sqrt{1-(u/c)^2}} \sim 1 + O((v/c)^2)$$

So in the "ultimate approximation" $\gamma = 1$

$$p = mu$$

Remark

The Book uses

relativistic mass $m(u) \equiv \gamma m_0$ rest mass

Implying a "relativistic" mass. In this way

$$p = m(u) \cdot u = \gamma m_0 u$$

$$E = m(u) c^2 = \gamma m_0 c^2$$

Almost no one uses this notation today. and I will not. In my notes

m always means m_0

Energy

$E = \gamma m c^2$ $\qquad \gamma = \frac{1}{\sqrt{1-(u/c)^2}} \simeq 1 + \frac{1}{2}\left(\frac{u}{c}\right)^2$

$$E \simeq \left[1 + \frac{1}{2}\left(\frac{u}{c}\right)^2\right] mc^2$$

$$E = \underbrace{mc^2}_{\text{rest energy}} + \underbrace{\frac{1}{2} m u^2}_{\substack{\text{non-rel} \\ \text{kinetic energy}}}$$

Makes sense to define the kinetic energy:

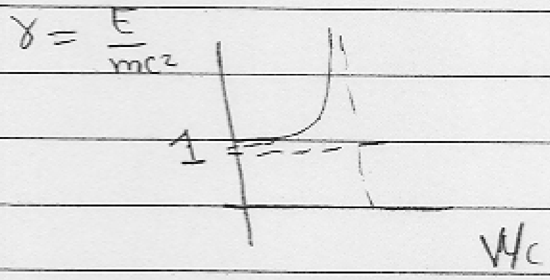
$$\boxed{K = E - mc^2} = (\text{Total energy}) - (\text{rest energy})$$

The Ultimate Speed:

- Energy can increase without bound. As you do more and more work

$$\frac{E}{mc^2} = \gamma \text{ goes to infinity}$$

Still $v/c < 1$



$$\gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

Thus It takes more and more work to increase the v/c even a tiny bit

Energy and Momentum Conservation

So Far used

① $F = \frac{dp}{dt}$ $\quad u = c^2\left(\frac{p}{E}\right)$ $\quad$ (not $\frac{p}{m}$ $\frac{1}{0}$)

② $dE = F\,dx$

To conclude $E = \gamma mc^2$ $\quad p = \gamma m u$:

Implicitly Used :

- momentum conservation
- Energy conservation

We will assume these to be true