Last Time

① Particles at Rest

- $E = mc^2$ mass is energy

② Important Numbers

A) • proton $m_p = 938 \frac{MeV}{c^2}$ • e^- $m_e = 0.511$ MeV/c^2

B) • $\underbrace{\quad}_{1\mathring{A} = 10^{-10} m}$ e^- gain energy by bringing together proton + electron

$$m_{Hydrogen} c^2 + \text{Energy gained} = m_p c^2 + m_e c^2$$

$$m_H c^2 = m_p c^2 + m_e c^2 + \underbrace{\text{Binding Energy}}_{\equiv -\text{Energy gained}}$$

③ Then we looked at moving particles

• $\longrightarrow$ $u \equiv$ particle velocity

$$E = \gamma_u mc^2 \qquad p = \gamma_u mu \qquad \gamma_u = \frac{1}{\sqrt{1-(u/c)^2}}$$

A) Studied the non-relativistic limit

$$E = \gamma_u mc^2 \qquad \gamma_u \simeq 1 + \frac{1}{2}\frac{u^2}{c^2}$$

$$\simeq mc^2 + \frac{1}{2}mv^2$$

rest energy — non-rel kinetic energy

$$p \simeq mv$$

B) Based on this defined

$$KE = E - mc^2 = \text{Energy} - \text{rest energy}$$

C) Now other important relations

$$u = c^2 \frac{p}{E} \qquad E = \sqrt{(mc^2)^2 + (cp)^2}$$

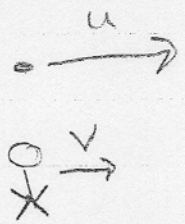
velocity is $\propto \frac{p}{E}$ — relation between energy and momentum

not $\frac{p}{m}$

D) For a massless particle (like light!)

$$E = cP \qquad u = c.$$

(4) Finally discussed lorentz transformations



- What does a right-mover see?

- Ans $u' = \dfrac{u_x - v}{1 - u_x v/c^2}$

Lowbrow answer, gets complicated fast

High brow: He measures a different energy and momentum

$$E' = \gamma_v E - \gamma_v \beta_v c p_x$$

$$c p_x' = -\gamma_v \beta_v E + \gamma_v c p_x$$

$$c p_y' = c p_y$$

$$c p_z' = c p_z$$

In one dimension

same matrix as for (ct, x)

$$\begin{pmatrix} E' \\ cp' \end{pmatrix} = \begin{pmatrix} \gamma_v & -\gamma_v \beta_v \\ -\gamma_v \beta_v & \gamma_v \end{pmatrix} \begin{pmatrix} E \\ cp \end{pmatrix}$$

So Far used

① $F = \frac{dp}{dt}$ $\qquad u = c^2 \frac{p}{E}$ $\qquad$ (not $\frac{p}{m}$!)

② $dE = F\,dx$

To conclude $\quad E = \gamma m c^2 \qquad p = \gamma m u$

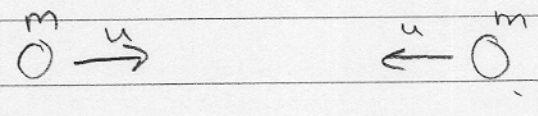
Implicitly Used

① - momentum conservation
② - Energy conservation

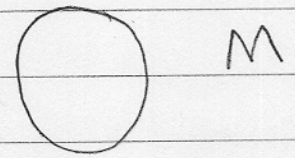
We will assume these to be true!

<u>Problem</u>: An Inelastic Collision

Before:

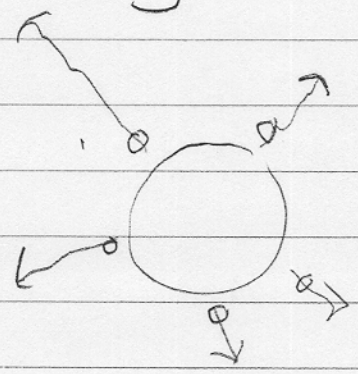


After:



- non-relativistically Kinetic energy gets converted into heat and is effectively lost

- Relativistically usually you make a blob

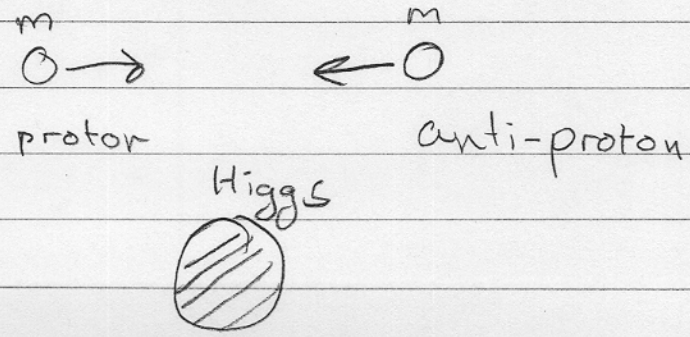


which spits off lots of other particles

- The mass of the final particles can be <u>much</u> greater than the incoming particle

<u>Problem</u>: The Higgs boson is puted to have a mass of $\sim 120\ GeV/c^2$. If a Higgs boson is formed during a proton, anti-proton collision, what is the minimum velocity which the proton and anti-proton must have to form this particle assuming nothing else is produced

<u>Sol</u> Before



Using E-consv:

- $$E_1 + E_2 = E_{final}$$

$$2E_p = m_{Higgs} c^2$$

energy of proton

$$E_p = \frac{m_H c^2}{2} \simeq 60 \text{ GeV}$$

- Compare this to the LHC Beam energy ~ 6 TeV. Its a lot (~100) less. Why?

What is the velocity?

$$E_p = \gamma m_p c^2 \implies \gamma = \frac{E}{m_p c^2} \simeq \frac{60 \text{ GeV}}{1 \text{ GeV}} \simeq 60$$

approximate prot mc

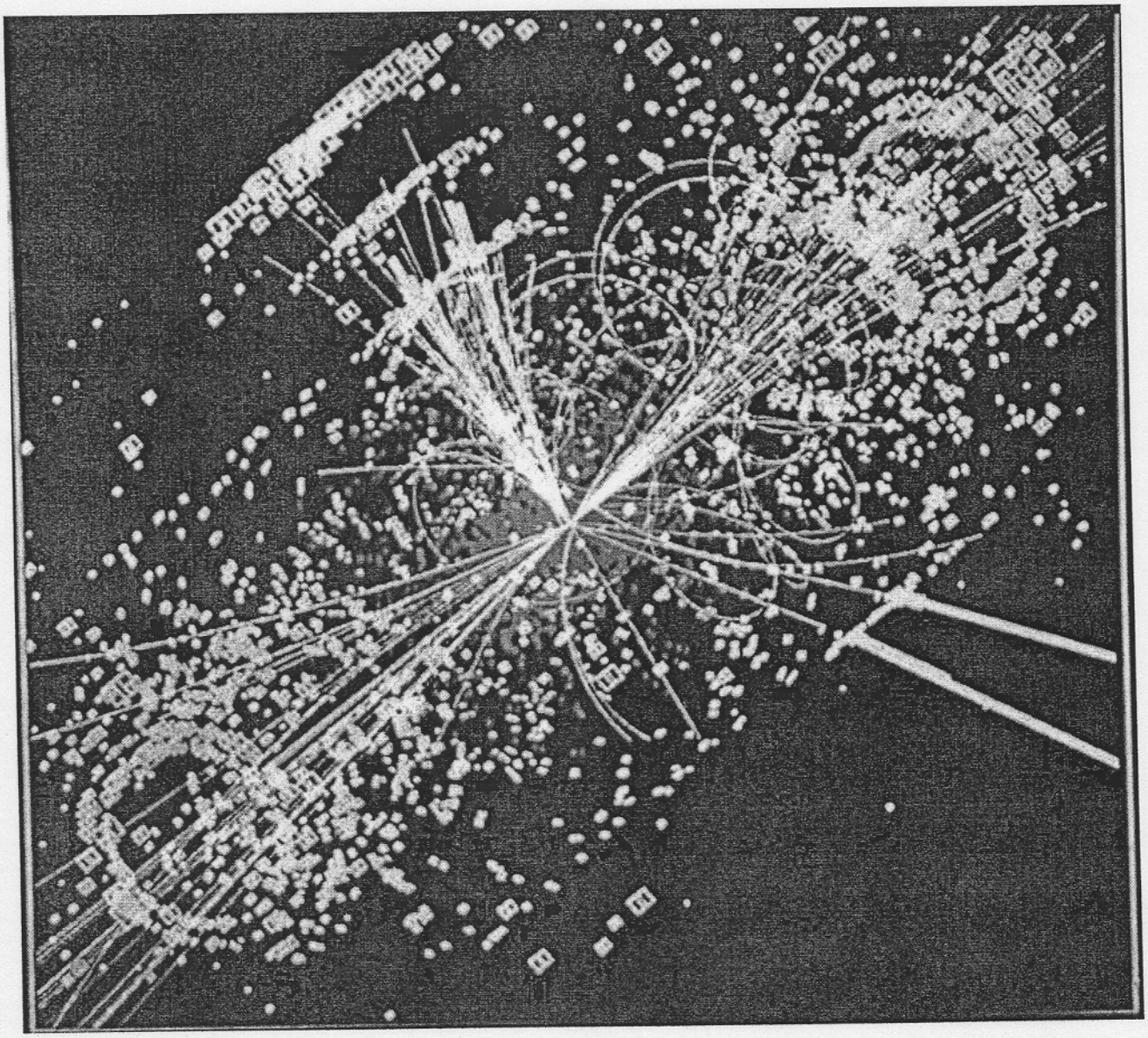
Now:

$$\gamma = \frac{1}{\sqrt{1-(u/c)^2}}$$

$$\frac{u^2}{c^2} = 1 - \frac{1}{\gamma^2}$$

$$\frac{u}{c} = \sqrt{1 - \frac{1}{\gamma^2}} \simeq \sqrt{1 - \frac{1}{60^2}}$$

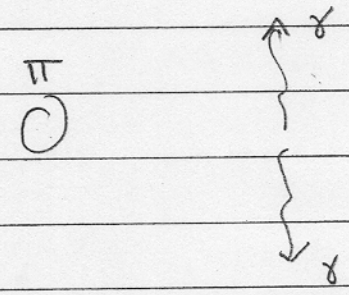
$$\frac{u}{c} \simeq 1 - 0.0004$$



Example

A π^0 decays as shown below into two photons. The mass of the pion

$\pi^0 \rightarrow 2\gamma$ $\qquad m_\pi = 140\ MeV/c^2$



What is the energy and momentum of the two photons

$$E_{before} = E_{after}$$

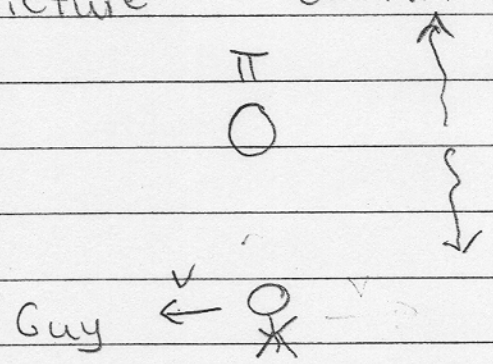
$$m_\pi c^2 = 2E_\gamma$$

$$E_\gamma = m_\pi c^2/2 \simeq 70\ MeV$$

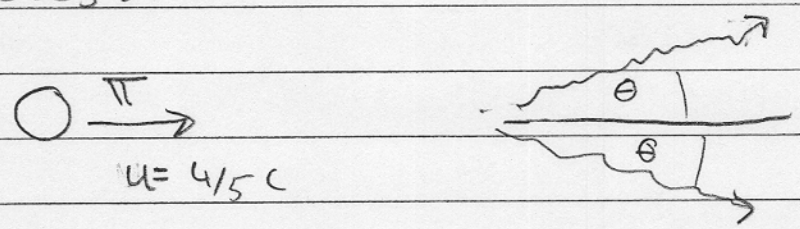
$$c p_{y\gamma} = E_\gamma \quad \text{so} \quad p_{y\gamma} = 70\ MeV/c \qquad p_x = 0\ p_z =$$

According to an observer moving to the <u>left</u> with speed 0.8c what is the energy and momentum of the light and π^0

Picture earth:



Guy sees:



Solution

The Energy of the pion:

$$E'_\pi = \gamma m_\pi c^2 \qquad \gamma = \frac{1}{\sqrt{1-(u/c)^2}} = \frac{5}{3} \qquad u = \frac{4c}{5}$$

$$E'_\pi = \frac{5}{3} m_\pi c^2$$

The momentum of

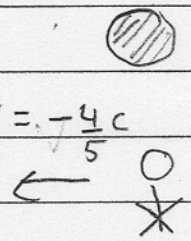
$$p'_{\pi,x} = \gamma m_\pi u$$

So

$$c p'_{\pi,x} = \gamma (m_\pi c^2) \frac{u}{c} = \frac{5}{3} (m_\pi c^2) \cdot \frac{4}{5} = \frac{4}{3} m_\pi c^2$$

Then lets use the lorentz transform to verify this result

$$E_\pi = m_\pi c^2 \quad cp_\pi = 0$$



$\gamma = 5/3 \qquad \beta = V/c$

$$E' = \gamma E - \gamma\beta cp = \frac{5}{3}(m_\pi c^2) - \frac{5}{3}\left(-\frac{4}{5}\right)(0)$$

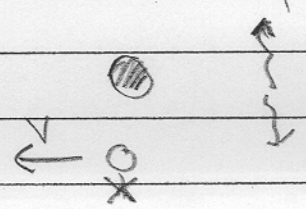
$$cp' = -\gamma\beta E + \gamma cp = -\left(\frac{5}{3}\right)\left(\frac{-4}{5}\right) m_\pi c^2 + \frac{5}{3}(0)$$

$$\left.\begin{aligned} E' &= \frac{5}{3} m_\pi c^2 \\ cp'_x &= +\frac{4}{3} m_\pi c^2 \end{aligned}\right\} \text{same as before}$$

$$cp'_y = 0$$

$$cp'_z = 0$$

Now Lets use the Lorentz transformation to figure out the energy and momentum of the two photons:



To the earth observer

$$E_\gamma = m_\pi c^2/2 \qquad cp_{y,\gamma} = \frac{m_\pi c^2}{2} \qquad cp_x = 0$$

Then according to left moving guy

$$E' = \gamma E - \gamma\beta\,(cp_x) = \frac{5}{3}\frac{m_\pi c^2}{2} - \frac{5}{3}\left(\frac{-4}{5}\right)\cdot 0$$

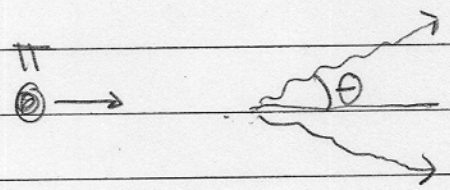
$$cp'_x = -\gamma\beta E + \gamma(cp_x) = -\frac{5}{3}\left(\frac{-4}{5}\right) m_\pi c^2 + \gamma\,(0)$$

$$cp'_y = cp_y$$

Or:

photons

$$E' = \frac{5}{3}\frac{m_\pi c^2}{2}$$



$$cp'_x = \frac{4}{3} m_\pi c^2/2$$

$$cp'_y = \pm m_\pi c^2/2$$

pion

$$E' = \frac{5}{3} m_\pi c^2$$

$$p'_x = \frac{4}{3} m_\pi c^2$$

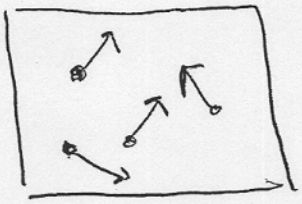
$$p'_y = 0$$

The opening angle $\theta = \tan^{-1}\frac{p_y}{p_x} = \tan^{-1}\frac{4/3\, m_\pi c^2/2}{m_\pi c^2/2} = \tan^{-1}\frac{4}{3} \simeq 53^\circ$

Quantum Nature of Light:

- History:

~1880 Mankind Understands what is temperature (Boltzmann)



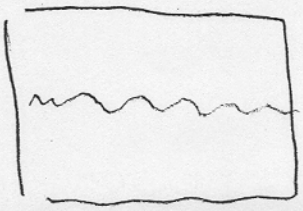
Answered how the molecules share the energy in the Box

$$P(E) \propto e^{-E/k_B T} = e^{-\frac{1}{2}mv^2/k_B T}$$

Boltzman Cons

Sometimes molecules move fast sometimes slow etc

Consider a String stretched accross the Box

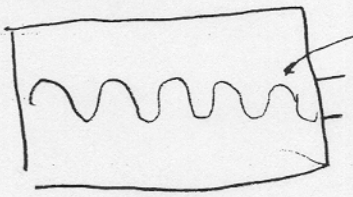


- The string will fluctuate
- People knew how to calculat the amplitude of fluctation

$$P(E) \propto e^{-E/k_BT} \qquad x = A\cos(kx - \omega t)$$

$$P \propto e^{-cA^2/k_BT} \qquad E_{\substack{in \\ string}} \propto A^2$$

Now consider a Box wich is red hot



- walls fluctuate, electrons fluctuate make E+M waves

- $\mathcal{E} = \mathcal{E}_0 \cos(Kx - \omega t)$

- Then according to classical rules

$$P(E) \propto e^{-E/k_BT} \propto e^{-c\mathcal{E}_0^2/k_BT}$$

- Should be able to predict the color, as a function of temperature
  red hot or blue hot etc
- The intensity vs. Temperature

$$I \propto \mathcal{E}_0^2$$

- They tried to calculate the color of the box vs. temperature, brightness of box vs. temperature

  - Lord Rayleigh

- It didn't work! "The UV-catastrophy"

<u>Planck's Answer:</u> The assumption that the amplitude of the wave $\mathcal{E}_0$ is continuous is wrong

- Light comes in discrete units (photons)

$$\boxed{E = hf}$$ energy of a single unit

$$P(n \text{ units of light}) \propto e^{-E_{TOT}/k_BT} \propto e^{-nh\nu/k_BT}$$

- works!

$$hc = 1240 \text{ eV nm} \qquad h = 6.6 \times 10^{-34} \text{ J}\cdot\text{s}$$

Problem: Calculate the Energy of A photon with wave length $\lambda = 620\ nm$ (red

$$E = hf = (hc)\left(\frac{f}{c}\right) = (hc)\left(\frac{1}{\lambda}\right) = \frac{1240\ eV\ nm}{62}$$

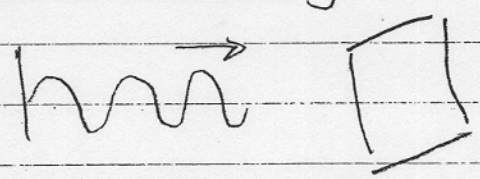
$$c = \lambda \nu$$

$$c = \lambda f$$

$$\frac{1}{\lambda} = \frac{f}{c}$$

$$E \simeq 2eV$$

What is Intensity:



- I = Energy which corosses per Area per tim

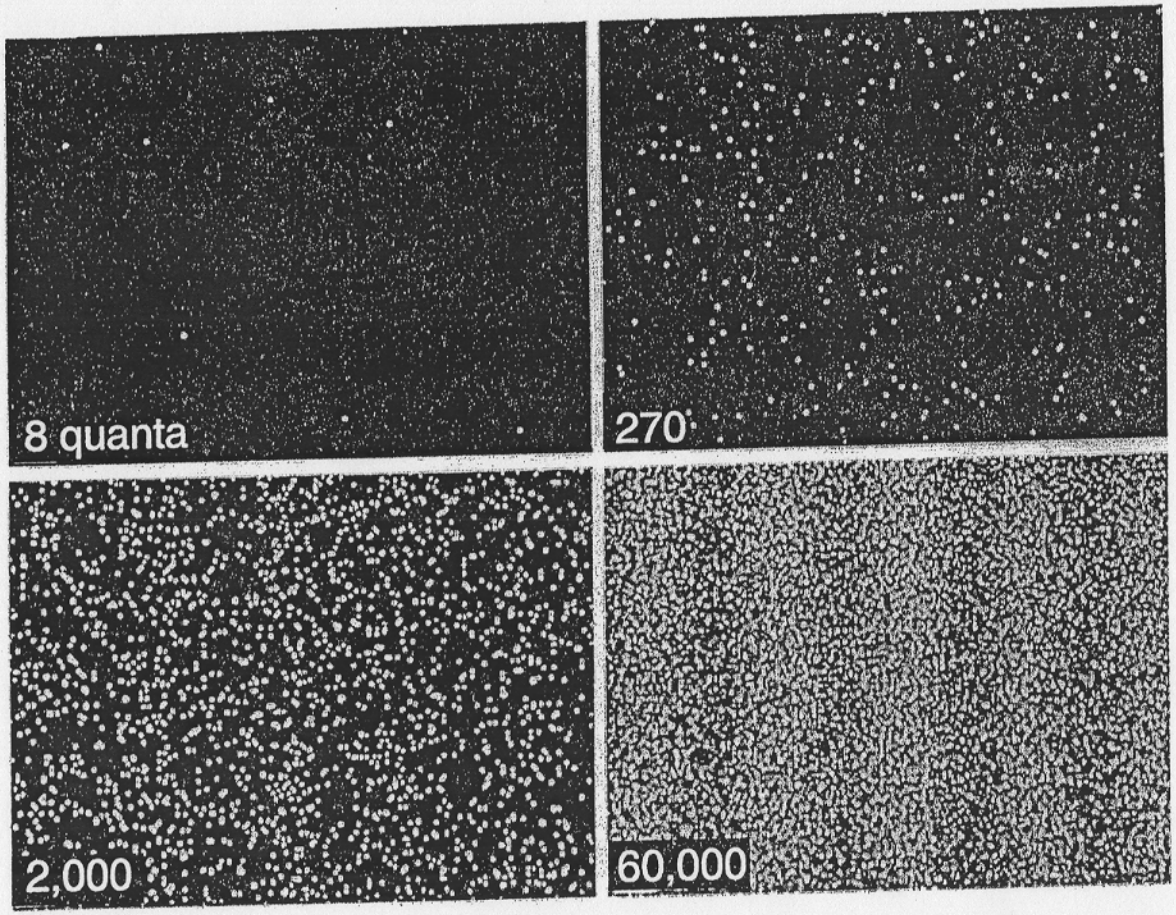
$$I = \langle |\vec{S}| \rangle = \langle \left| \frac{\vec{\mathcal{E}} \times \vec{B}}{\mu_0} \right| \rangle = \frac{\sqrt{\mu_0 \varepsilon_0}}{\mu_0} \mathcal{E}^2 = \sqrt{\frac{\varepsilon_0}{\mu_0}} \langle \mathcal{E}$$

$$B = \frac{\mathcal{E}}{c} \leftarrow \text{for a wave} \qquad c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

- $$I = \underbrace{(hf)}_{\text{energy of a photon}} \times \langle (\text{number of photons which cross per unit area per unit tim}$$

$$\boxed{\langle \mathcal{E}^2 \rangle \propto \langle \#\ \text{of photons per area per time} \rangle \propto I}$$

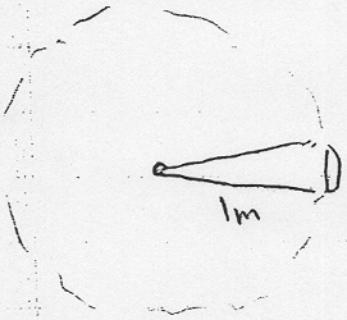
Photons arriving one-by-one



Electrons, really, but you get the idea. . . From A. Tonomura, *Electron Holography* (Springer-Verlag, 1993), p. 14.

Problem:

Calculate the number of photons which are hitting the page per unit time. Take $\lambda = 620\,nm$, take a $100\,W$ light source of one meter from the page which is $10\,cm \times 10\,cm$



$$\text{Energy crossing page per time} = \mathcal{P}_{source} \frac{A_{page}}{(4\pi R^2)}$$

$$= 100\,W \cdot \frac{(0.1\,m)}{(4\pi\, 1\,m^2)}$$

$$= 0.08\,W = 0.08\,\frac{J}{s}$$

$$1\,J = 0.6 \times 10^{19}\,eV$$

So

$$I = h f \quad \langle \text{number of photons / sec} \rangle$$

$$0.08 \times 0.6 \times 10^{19}\,\frac{eV}{s} = (2\,eV) \quad \langle \#/s \rangle$$

$$\boxed{2.4 \times 10^{17}\,\frac{1}{s} = \#\text{ per sec}}$$

Enormous

If you want to see the quantum nature of Light we need # of photons small

$$hf \ (\#\text{ per sec}) = \langle I \rangle \cdot \text{Area of Detector}$$

$$\#\text{ per sec} = \frac{\langle I \rangle \text{ Area of Detector}}{hf}$$

$$\# = \frac{\langle I \rangle \cdot (A)\ \Delta t}{hf}$$

Need

1. Low intensities - Hard to measure
2. High Frequencies (UV + X-rays)
3. Fast Detector
4. Small Detector Size

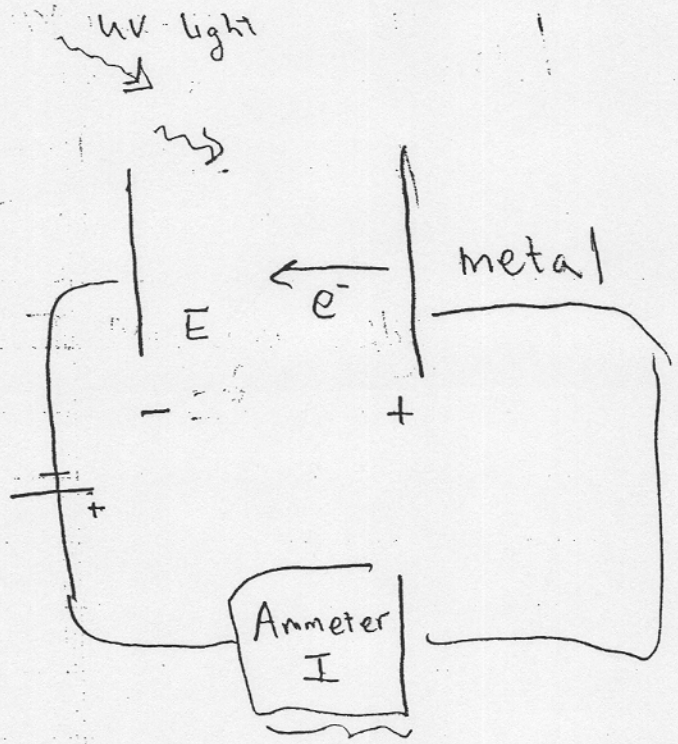
Suppose we could make a detector the size of an atom

$$r = 10^{-10}\ m \qquad \text{Area} = 10^{-20}\ m^2$$

Then detector would see the "graininess" of light

The photo electric effect:

Hertz (1880)
Leonard (~1900)



A challenge at the time

- Current was proportional to Intensity (
  - This was expected

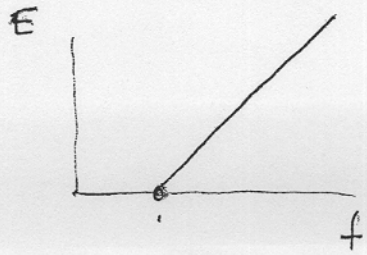
<u>Electron Energy</u>

- As you increase the voltage, eventually the current stops

$$KE = \frac{1}{2} m v_{max}^2 = |e| V_s$$

<u>Find</u>

1. Energy proportional to frequency,
2. independent of Intensity



Classical Expectations:

- Force By light on Electron $\propto$ Electric Field $\propto \sqrt{I}$

- Independent of Frequency

Einstein:

- Thought Deeply about Planck

- realizes that a metal atom will see one photon, emphasizes that Laws of E+M could be different. Maxwell's Equations work over long time scales lots of photons

Photon Energy: $E = hf$

Assume Photon gives all its energy to electron

$K_e = hf$ – (energy needed to rip electron from atom)

$$K_e = \underbrace{h}_{\text{universal constant}} f - \underbrace{\phi}_{\text{Work Function}}$$

Work Function varies from metal to metal

$\phi_{Na} = 2.28\ eV$ $\phi_{Fe} = 4.5\ eV$

Problem

Determine the velocity of the electron when it is exposed by light of wavelength (141 nm) (Deep UV) frequency (2.1×10^{15} Hz). Assume the work function of iron is 4.5eV, what is the stopping voltage

Solution

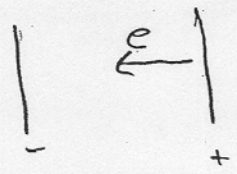
$$K = hf - \phi$$

$$K = (hc)\frac{f}{c} - \phi \qquad c = \lambda f \quad \text{so} \quad \frac{1}{\lambda} = \frac{f}{c}$$

$$K = 1240\,eV\,nm\left(\frac{1}{\lambda}\right) - 4.5\,eV$$

$$K = 1240\,eV\,m\left(\frac{1}{141\,nm}\right) - 4.5\,eV = 4.3\,eV$$

Then:



$$e|V_s| = \text{Kinetic Energy}$$

$$\cancel{e}|V_s| = 4.3\,\cancel{e}V$$

$$|V_s| = 4.3\,V$$

To find the electron velocity

$$\frac{1}{2}m_e v^2 = K$$

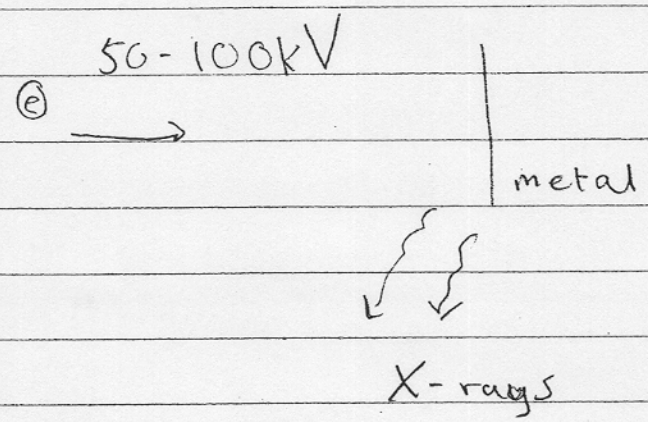
$$\frac{1}{2} m_e c^2 \left(\frac{v}{c}\right)^2 = K$$

$$\frac{v}{c} = \sqrt{\frac{2K}{m_e c^2}} = \sqrt{\frac{2(4.3\,eV)}{0.5\,MeV}} = 0.0041$$

very non-relativistic!

$$v = 1.2 \times 10^6\ m/s$$

X-rays



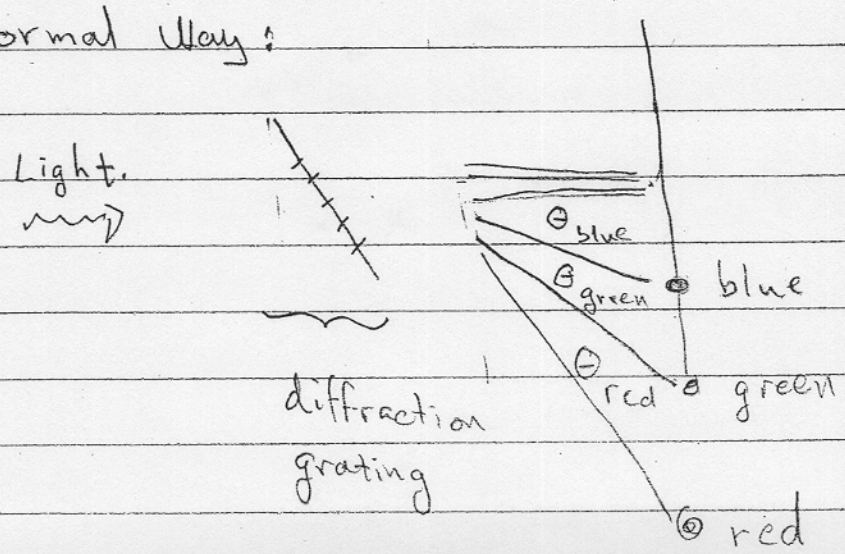
- Very high frequency $\lambda_X \sim 10^{-10}$ m

  – Compare $\lambda_{visible} \sim 6000 \times 10^{-10}$ m

- $E = hf$ is a big step! This light is not continuous on human scales

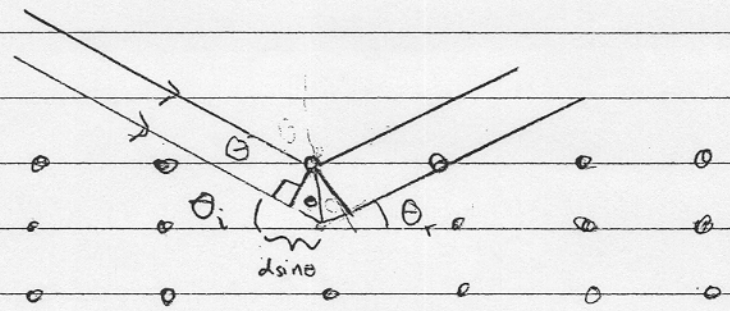
- Problem how to measure the wavelength?

Normal Way:



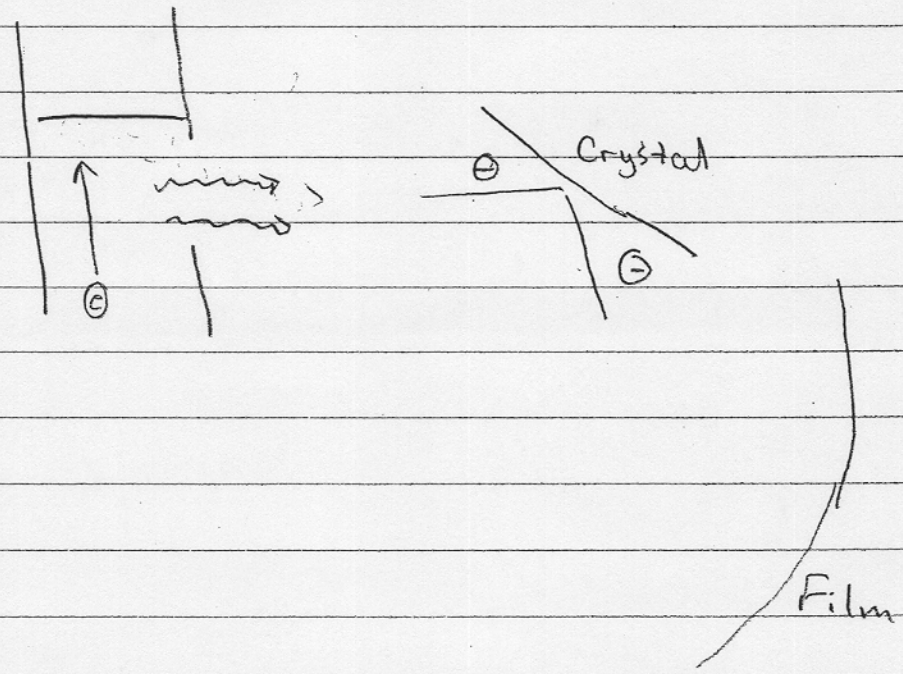
How to make a diffraction grating where the spacing is $\sim \lambda$.

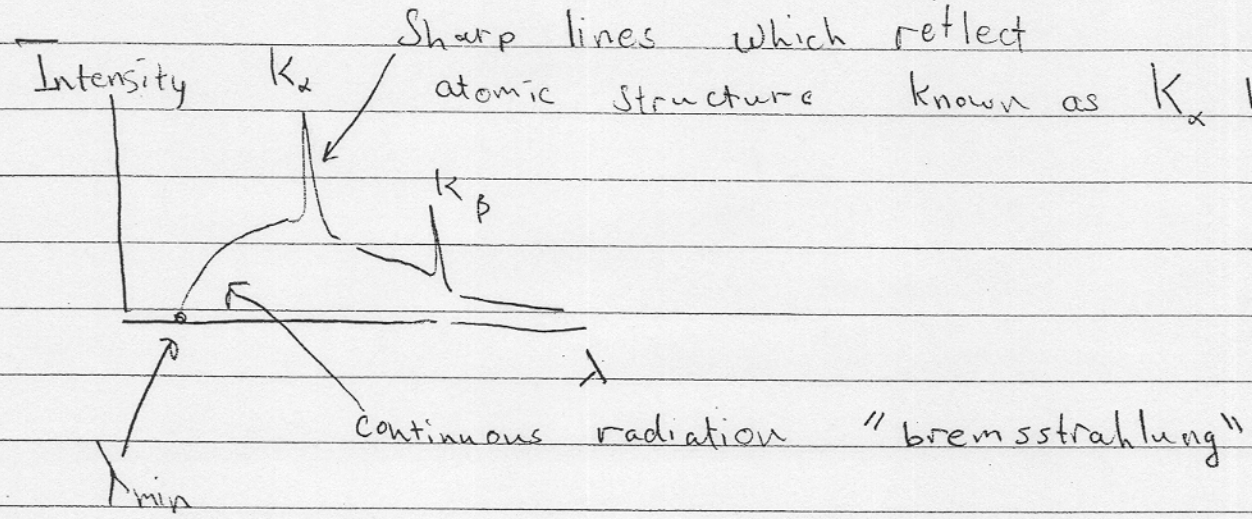
Answer: Make a diffraction grating out of a crystal (Bragg, von Laue)



$$2d\sin\theta = n\lambda \qquad n = 1, 2, 3, 4, \ldots$$

<u>Result</u>



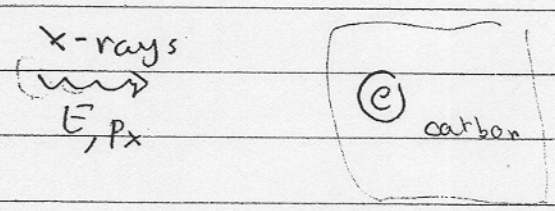


λ_{min} is determined when all of then energy of the electron is converted to a single photo

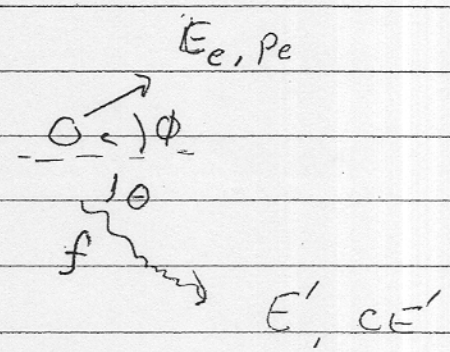
$$eV = hf = h\frac{c}{\lambda_{min}}$$

The Compton Process (1922)

Before:



After:



Use relativistic energy and momentum conservation and

$$E = hf$$

Energy

(1) $E + m_e c^2 = E' + E_e$

Momentum

(2) $p = p' \cos\theta + p_e \cos\phi$

(3) $0 = -p' \sin\theta + p_e \sin\phi$

Also have: $E = cp$, $E' = cp'$, $E_e = \sqrt{(cp_e)^2 + m_e c}$

Algebra: use the 1 equation to eliminate p_e, ϕ

$p' \sin\theta = p_e \sin\phi$ (2)

$(p - p' \cos\theta) = p_e \cos\phi$ (3)

Squaring both equations and add:

$p^2 - 2pp' \cos\theta + p'^2 \cos^2\theta = p_e^2 \cos^2\phi$

$p'^2 \sin^2\theta = p_e^2 \sin^2\phi$

$p^2 - 2pp' \cos\theta + p'^2 = p_e^2$ $\Leftarrow$ so far used one equ to elim.

Now use E-consve

$$(cp - cp') + m_e c^2 = E_e$$

$$\left[(cp - cp') + m_e c^2\right]^2 = E_e^2$$

$$(cp - cp')^2 + 2(cp - cp')(m_e c^2) + (m_e c^2)^2 = E_e^2$$

$$(cp - cp')^2 + 2(cp - cp')\, m_e c^2 = E_e^2 - m_e c^2$$

$$(cp - cp')^2 + 2(cp - cp')\, m_e c^2 = (cp_e)^2$$

$$(p - p')^2 + 2(p - p')\, m_e c = p_e^2$$

Putting in:

$$p^2 - 2pp'\cos\theta + p'^2 = (p - p')^2 + 2(p - p')\, m_e c$$

Working:

$$\frac{1}{m_e c}(1 - \cos\theta) = \frac{1}{p'} - \frac{1}{p}$$

Now assume: $p = h\nu$ $\quad p = h\nu'$

$$\frac{1}{m_e c}(1 - \cos\theta) = \frac{1}{h\nu'} - \frac{1}{h\nu}$$

Or

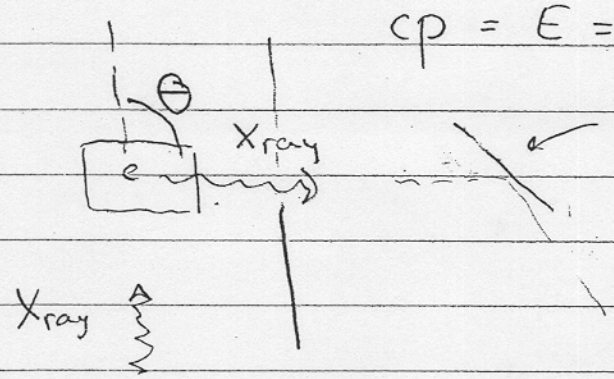
$$\frac{h}{m_e c}(1-\cos\theta) = \frac{c}{\nu'} - \frac{c}{\nu}$$

$$\frac{h}{m_e c}(1-\cos\theta) = \lambda' - \lambda$$

Important Points:

(1) • Used, Relativity and Used

$cp = E = hf$, and worked!



"Bragg's Crystal

(2) $\frac{h}{m_e c}$ = The compton wavelength

$$\frac{hc}{m_e c^2} = \frac{1240\,eV\,nm}{.511\times10^3\,eV} = \underbrace{0.0024\,nm}_{\simeq \frac{1}{137}\text{ times the size of an atom}} \equiv \lambda_c$$

The shift in wave length is this order of magnitude

$$\lambda_c(1-\cos\theta) = \lambda' - \lambda \qquad \text{for } \theta = 90^\circ$$

$$\lambda_c = \lambda' - \lambda$$

For optical light $\lambda = 500\,nm$ and we have

$$\lambda' = \lambda + \lambda_c$$

$$\lambda' = 500\,nm + .0024\,nm \simeq 500\,nm$$

Thus it was essential to use X-rays with wave length of order

$$\lambda \sim \lambda_c \sim 0.0024\,nm$$

(3) What was the energy of the recoiling electrons take $\lambda = \lambda_c$ and $\theta \simeq 90^\circ$

$$E + m_e c^2 = E' + E_e$$

$$\lambda_c (1 - \cancel{\cos\theta}^{\,0}) = \lambda' - \lambda$$

$$\lambda_c = \lambda' - \underbrace{\lambda}_{\lambda_c}$$

$$2\lambda_c = \lambda'$$

$$E = hf = h\frac{c}{\lambda} = \frac{hc}{\frac{hc}{m_e c^2}} = m_e c^2$$

$$E' = hf' = h\frac{c}{\lambda'} = \frac{hc}{2hc/m_e c^2} = \frac{m_e c^2}{2}$$

So :

$$m_e c^2 + m_e c^2 = \frac{m_e c^2}{2} + E_e$$

$$\frac{3}{2} m_e c^2 = E_e = \gamma m_e c^2$$

$$\frac{3}{2} = \gamma \qquad \beta = \frac{v}{c}$$

$$\frac{3}{2} = \frac{1}{\sqrt{1-\beta^2}} \qquad \beta \approx 0.74$$

electrons are pretty relativistic.

Q: How important is the binding energy