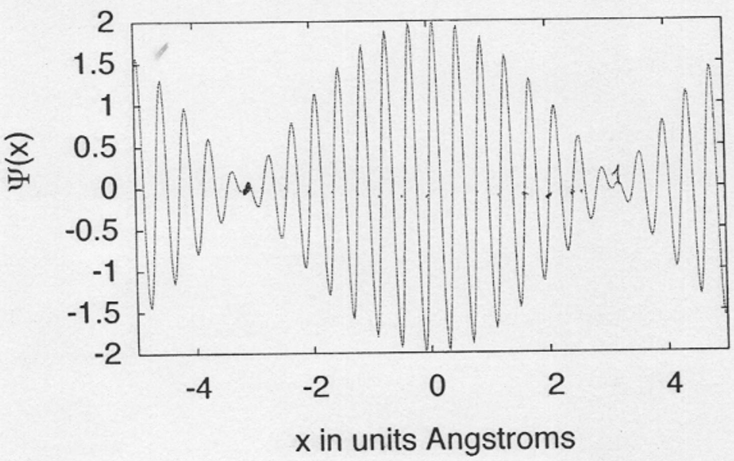
Problems: due 11/4

3.19, 3.28, 3.30, 3.31, 5.2 (only a,b,c) – **graded**, 5.7, 5.10, 5.11

1. **(graded)** Consider the graph shown below wich shows the electron wave function at a specific time which is not subject to an external potential, i.e. $V(x) = 0$

   (a) Estimate from the graph below the average momentum and energy of the electron.

   (b) Estimate the uncertainty in this average momentum.



2. **Complex Numbers** (a) Show $1 + i = \sqrt{2}e^{i\pi/4}$ and $1 - i = \sqrt{2}e^{-i\pi/4}$ (b) Show $|e^{ikx}|^2 = 1$ (c) Show $|e^{ik_1x} + e^{ik_2x}|^2 = 2(1 + \cos(\Delta k\, x))$ with $\Delta k = k_1 - k_2$ (d) A general wave funtion is $\Psi(x) = R(x) + iI(x)$ where $R(x)$ and $I(x)$ are real functions. Show that $|\Psi|^2$ is positive. (e) A general wave function is $\Psi(x) = A(x)e^{i\phi(x)}$ where $A(x)$ and $\phi(x)$ are real functions, show that $|\Psi(x)|^2 = A(x)^2 = R(x)^2 + I(x)^2$.

Answers

3.19: 15 keV, 0.1 MeV hard X-ray (5.2) (5.7) 0.196 (5.10) $A = \sqrt{2/a}$ (5.11) $\bar{x} = 0$ and $\overline{x^2} = a^2(0.07)$ (1.) 31 keV/c, $E = 1.8$ keV. $\Delta p \sim 3.1$ keV/c.

Basic notions of wave functions

1. DeBroglie says that the momentum is related to the wavelength

$$p = \frac{h}{\lambda} = \hbar \frac{2\pi}{\lambda} = \hbar k \tag{1}$$

2. Similarly the frequency determines energy

$$E = \hbar\omega \qquad \omega = 2\pi\nu \tag{2}$$

   where ν is the frequency.

3. If the typical size of the wave function is Δx then the typical spread is in the momentum Δp is determined by the uncertainty relation

$$\Delta x \Delta p \gtrsim \hbar/2 \tag{3}$$

4. Similarly if the typical duration of a wave pulse (of e.g. sound, E&M, or electron wave) is Δt then its frequency ω is only determined to within $1/\Delta t$. In quantum mechanics this is written

$$\Delta t \Delta\omega \sim \frac{1}{2} \qquad \text{or} \qquad \Delta t \Delta E \gtrsim \hbar/2$$

   i.e. if something is observed for a short period of time its energy can not be precisely known

5. In general an attractive potential energy tends to localize (make smaller) the particles wave function. As the particle is localized the kinetic energy increases. The balance determines the typical extent of the wave function (or the size of the object).

Wave packets

1. A general wave can be written as a sum of sin's and cos's. For a general wave then there is not one momentum and energy associated with the particle but a range of momenta and energies characterized by $\Delta\omega$ and Δk

2. Consider the addition of two waves

$$\Psi_1(x,t) = \sin(k_1 x - \omega_1 t) \qquad \Psi_2(x,t) = \sin(k_2 x - \omega_2 t)$$

The waves have a certain average frequency (energy) $\bar{\omega} = (\omega_1 + \omega_2)/2$ and a frequency spread $\Delta\omega = \omega_1 - \omega_2$. Similarly the two waves have an average wave number (momentum) $\bar{k} = k_1 + k_2$ and $\Delta k = k_1 - k_2$. The sum of the two waves is

$$\Psi_1 + \Psi_2 = 2 \underbrace{\sin(\bar{k}x - \bar{\omega}t)}_{\text{Carrier wave}} \underbrace{\cos(\frac{\Delta k}{2}x - \frac{\Delta\omega}{2}t)}_{\text{Envelope Wave}}$$

When we talk about the energy and momentum of a wave we are really talking about the average momentum (wave number k) and average energy (angular frequency ω).

3. The speed of the envelope is group velocity

$$v_g = \frac{\Delta\omega}{\Delta k} = \frac{d\omega}{dk} = \frac{dE}{dp} \tag{4}$$

4. The spatial extent of the wave packet is of order the wavelength of the envelope

$$\Delta x \sim \frac{1}{\Delta k} \qquad \Delta x \Delta k \sim 1 \tag{5}$$

In quantum mechanics this becomes $\Delta x \Delta p \sim \hbar$

5. The temporal extent of the wave packet is of order the period of the evnvelop

$$\Delta t \sim \frac{1}{\Delta\omega} \qquad \Delta t \Delta\omega \sim 1 \tag{6}$$

In quantum mechanics this becomes $\Delta t \Delta E \sim \hbar$

6. The same analysis can be done using complex exponentials. Consider the addition of two waves

$$\Psi(x,t) = e^{-i\omega_1 t + ik_1 x} + e^{-i\omega_2 t + ik_2 x} \tag{7}$$

You should be able to show that

$$\Psi(x,t) = \underbrace{e^{-i\bar{\omega}t + i\bar{k}x}}_{\text{carrier}} \underbrace{2\cos\left(\frac{\Delta\omega}{2}t - \frac{\Delta k}{2}x\right)}_{\text{envelope}} \tag{8}$$

Wavefunctions

1. The electron wave function squared $|\Psi(x),t|^2 = \mathrm{P}(x,t)$ is *a probability per unit length* to find the particle at time t. The the probability $d\mathscr{P}$ to find a particle between x and $x + dx$ at time t

$$d\mathscr{P} = \mathrm{P}(x,t)dx = |\Psi(x,t)|^2 dx \tag{9}$$

2. The most likely location at time t may be found by maximizing the probability density $\mathrm{P}(x,t)$

3. The electron must be somewhere so

$$\int_{-\infty}^{\infty} dx\, |\Psi(x,t)|^2 = 1 \tag{10}$$

4. The average position at time t

$$\overline{x} = \int_{-\infty}^{\infty} dx\, x\, |\Psi(x,t)|^2 \tag{11}$$

5. The average position squared at time t is

$$\overline{x^2} = \int dx\, x^2\, |\Psi(x,t)|^2 \tag{12}$$

6. The uncertainty squared in position $(\Delta x)^2$ is defined to be

$$(\Delta x)^2 \equiv \overline{x^2} - \overline{x}^2 = \overline{(x-\overline{x})^2} \tag{13}$$

   This is also known as the standard deviation squared, or the spread. If the average position is zero $\bar{x}$ then $(\Delta x) \equiv \sqrt{\overline{x^2}}$.

Problem 1

The graph is a sum of two sinusoidal waves:

$$\psi = \sin(k_1 x) + \sin(k_2 x)$$

$$= \sin\left(\left(\bar{k} + \frac{\Delta k}{2}\right)x\right) + \sin\left(\left(\bar{k} - \frac{\Delta k}{2}\right)x\right)$$

$$= 2\sin(\bar{k}x)\cos\left(\frac{\Delta k}{2}x\right)$$

Counting the wavelength of the rapid variation:

$$\bar{k} = \frac{2\pi}{\lambda} \qquad \lambda \sim \frac{6\,\text{Å}}{15} \approx 0.4\,\text{Å}$$

15 wavelengths over 6 Å

$$\hbar k = \frac{2\pi}{0.4\,\text{Å}}\,\hbar c\,\frac{1}{c}$$

$$\hbar k = \frac{2\pi}{0.4\,\text{Å}}\,(1970\ \text{eV Å})\,\frac{1}{c}$$

$$\hbar k \sim 31\,\frac{\text{keV}}{c}$$

Then

$$\frac{\Delta k}{2} \simeq \frac{2\pi}{\text{wave length of envelope}}$$

$$\Delta k \simeq \frac{4\pi}{2 \cdot 6 \text{Å}}$$

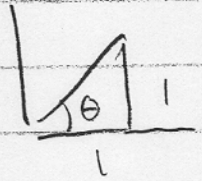
$$\Delta p = \hbar \Delta k = \frac{\hbar c}{c} \Delta k = 1970 \frac{\text{eVÅ}}{c} \cdot \frac{4\pi}{12 \text{Å}}$$

$$\boxed{\Delta p \sim 2 \frac{\text{keV}}{c}}$$

$$E = \frac{(cp)^2}{2mc^2} \sim \frac{(31 \text{ keV})^2}{2(511 \text{ keV})} \simeq 0.94 \text{ keV}$$

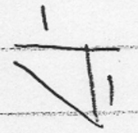
Complex Numbers

a) $z = 1 + i = \sqrt{2}\, e^{i\pi/4}$



$\sqrt{2} = \sqrt{1^2 + 1^2}$ $\quad \theta = 45^\circ$

b) $z = e^{-i\pi/4} \cdot \sqrt{2}$



$\theta \simeq -45^\circ$ $\quad r = \sqrt{1^2 + 1^2}$

c) $|e^{+ikx}|^2 = e^{-ikx} e^{+ikx} = e^0 = 1$

The modulus of a phase is one

d) $|e^{ik_1x} + e^{+ik_2x}|^2 = (e^{-ik_1x} + e^{-ik_2x})(e^{+ik_1x} + e^{+ik_2x})$

$= e^{-ik_1x} e^{+ik_1x} + e^{-ik_2x} e^{-ik_2x} + e^{-ik_2x} e^{+ik_1x} +$

$+ e^{-ik_1x} e^{+ik_2x}$

$= 1 + 1 + e^{i\Delta kx} + e^{-i\Delta kx}$

$= 2 + 2\cos(\Delta kx)$

d) $\psi = R + iI$

$$|\psi|^2 = \underbrace{(R - iI)}_{\psi^*}\underbrace{(R + iI)}_{\psi} = R^2 + I^2$$

e) $\psi = A e^{i\phi}$

$$\psi^* \psi = (A e^{-i\phi})(A e^{i\phi}) = A^2$$

Problem 3.2

$p = h/\lambda$

$$K = \frac{p^2}{2m} = \frac{h^2}{2m\lambda^2} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{(1240 \text{ eV nm})^2}{2(511000 \text{ eV})(589\text{n}}$$

$$K = 4.3 \times 10^{-5} \text{ eV}$$

This is quite small

Prob 3.19

$\lambda \sim 0.1 \text{ Å}$

a) $$K = \frac{p^2}{2m} = \frac{(hc)^2}{2mc^2\lambda^2} = \frac{(1240 \text{ eV nm})^2}{2(511000 \text{ eV } (0.01 \text{nm})^2)}$$

$$K = 15 \text{ keV}$$

(b) $E_\gamma \simeq \frac{hc}{\lambda} = \frac{1240 \text{ eV nm}}{0.01 \text{ nm}} = 0.1 \text{ MeV}$

hard x-rays

(c) Its much easier to make a beam of keV electrons

3.28

$p \sim \frac{\hbar}{L}$ $\qquad L \sim 1 \text{ Å} \sim 2a_0$

a) $K \sim \frac{p^2}{2m} \sim \frac{\hbar^2}{2mL^2} \sim 3.4 \text{ eV}$ $\qquad$ yes this is consistent with atomic bindings

b) $K \sim \frac{\hbar^2}{2mL^2}$ $\qquad L \sim 10 \text{ fm}$

$K \sim \frac{(197 \text{ MeV fm})^2}{2(0.5 \text{ MeV})(10 \text{ fm})^2} \sim 389 \text{ MeV}$

This is not consistent w nuclear binding energies BE ~ 8 MeV

c) For protons $K \sim \frac{(197 \text{ MeV fm})^2}{2(938 \text{ MeV})(10 \text{ fm})^2} \simeq 0.19$ MeV

this is consistent w nuclear binding

3.30

$$\Delta E_1 \sim \frac{\hbar}{\Delta t_1} \sim \frac{\hbar c}{c\Delta t_1} \sim \frac{197\ eV\,nm}{(3\times10^8 \frac{m}{s})(1.2\times10^{-8} s)} \simeq 5.4\times10^{-8}$$

$$\Delta E_2 \sim \frac{\hbar}{\Delta t_2} \sim \frac{\hbar c}{c\Delta t_2} \sim \frac{197 eV\,nm}{(3\times10^8 m/s)(2.3\times10^{-8} s)} \simeq 2.8\times10^{-8}\ eV$$

$\Delta E_{TOT} \sim 8.2\times10^{-8}\ eV$ ~ about twice the "book" estimate

3.31

$$E = \sqrt{(cp)^2 + (mc^2)^2}$$

$$V_g = \frac{\partial E}{\partial p} = \frac{1}{\cancel{2}} \frac{\cancel{2}\, c^2 p}{\sqrt{(cp)^2 + (mc^2)^2}} = c^2 \frac{p}{E} \checkmark$$

I.e. $p = \gamma m v \qquad E = \gamma mc^2$

$c^2 p = \gamma mc^2 v \qquad E = \gamma mc^2$

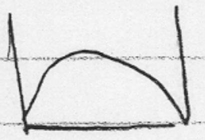
So $\frac{c^2 p}{E} = \frac{\gamma \cancel{mc^2} v}{\gamma \cancel{mc^2}} = v$ (your supposed to know $\frac{c^2 p}{E} = v$) anyway

5.7

$$\Psi(x,t) = \sqrt{\frac{2}{a}}\cos\left(\frac{\pi x}{a}\right) e^{-iEt/\hbar}$$

norm of a phase is one

$$|\Psi|^2 = \frac{2}{a}\cos^2\left(\frac{\pi x}{a}\right)\left|e^{-iEt/\hbar}\right|^2$$



$$= \frac{2}{a}\cos^2\left(\frac{\pi x}{a}\right)$$

So

$$P = \int_{(a/2 - a/3)}^{a/2} |\Psi|^2 dx = \int_{a/6}^{a/2} \frac{dx}{a}\, 2\cos^2\left(\frac{\pi x}{a}\right)$$

note the limit of integration the end of box is at a/2 while we want to star[t] $\frac{1}{3}$a fr[om] end

let $u = x/a$

$$P = \int_{1/6}^{1/2} du\, 2\cos^2(\pi u) = \int_{1/6}^{1/2} du\, \cancel{2}\left[\frac{+\cos(2\pi u) + 1}{\cancel{4}}\right]$$

$$P = \int_{1/6}^{1/2} du\left[+\cos(2\pi u) + 1\right]$$

$$P = \frac{1}{3} + \frac{1}{2\pi}\sin(2\pi u)\Big|_{1/6}^{1/2}$$

$$P = \frac{1}{3} - \frac{1}{2\pi}\sin\left(\frac{\pi}{3}\right)$$

$$P = \frac{1}{3} - \frac{1}{2\pi}\cdot\frac{\sqrt{3}}{2} = \boxed{0.1955 = P}$$

5.10

$$\Psi(x,t) = A \sin\left(\frac{2\pi x}{a}\right) e^{-iEt/\hbar}$$

$$|\Psi(x,t)|^2 = A^2 \sin^2\left(\frac{2\pi x}{a}\right)$$

So

$$\int_{-a/2}^{a/2} |\Psi(x,t)|^2 dx = A^2 \int_{-a/2}^{a/2} \sin^2\left(\frac{\pi x}{a}\right) dx = 1$$

$$= A^2 a \left\langle \sin^2\left(\frac{\pi x}{a}\right) \right\rangle = 1$$

Length of box average height of $\sin^2$

$\langle \sin^2 \rangle = \frac{1}{2}$ so

a) $A^2 a \frac{1}{2} = 1 \Rightarrow A = \sqrt{\frac{2}{a}}$

(b) Its the same $\Leftrightarrow$ no deep reason I can think of

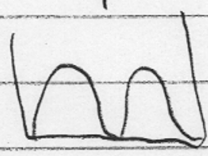
P11

$$\psi(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{2\pi x}{a}\right) e^{-iEt/\hbar}$$

$$|\psi|^2 = \frac{2}{a} \sin^2\left(\frac{2\pi x}{a}\right)$$

$$\bar{x} = \int_{-a/2}^{a/2} dx \; \underset{\uparrow}{x} \; \underset{\uparrow}{\frac{2}{a} \sin^2\left(\frac{2\pi x}{a}\right)} = 0$$

odd even



$$\overline{x^2} = \int_{-a/2}^{a/2} dx \; x^2 \sin^2\left(\frac{2\pi x}{a}\right) \cdot \frac{2}{a}$$

$$\overline{x^2} = a^2 \int_{-a/2}^{a/2} \frac{dx}{a} \left(\frac{x^2}{a^2}\right) \sin^2\left(\frac{2\pi x}{a}\right) \cdot 2$$ Let $u = x/a$

$$x^2 = a^2 \int_{-1/2}^{1/2} du \; u^2 \sin^2(2\pi u) \cdot 2$$

By parts twice see next pager

$$x^2 = a^2 \left(\frac{1}{12} - \frac{1}{8\pi^2}\right) \simeq a^2 (0.07)$$

$$\boxed{\sqrt{\overline{x^2}} = a\,(0.265)}$$

I would give you these on the test, but if you want to be actually good at this you should do it

$$I = \int du\, u^2 \sin^2(2\pi u) \cdot 2$$

$$= \cancel{2} \int_{-1/2}^{1/2} du\, u^2 \left[\frac{-\cos(4\pi u)+1}{\cancel{2}}\right]$$

$$= \int_{-1/2}^{1/2} du\, u^2 \left[1 - \cos 4\pi u\right]$$

$$= \frac{u^3}{3}\Big|_{-1/2}^{1/2} + \int_{-1/2}^{1/2} du\, -u^2 \cos \pi u$$

$$= 2 \cdot \frac{1}{24} + 1 \int_{-1/2}^{1/2} du\, u^2 \frac{1}{(4\pi)^2} \frac{d^2}{du^2} \cos(4\pi u)$$

$$= \frac{1}{12} + \cancel{u^2 \frac{d}{du} \cos(4\pi u)\Big|_{-1/2}^{1/2}}^{\,0} - \int_{-1/2}^{1/2} 2u \frac{1}{(4\pi)^2} \frac{d}{du} \cos(4\pi u)$$

$$= \frac{1}{12} - \frac{1}{(4\pi)^2}\, 2u \cos 4\pi u \Big|_{-1/2}^{+1/2} + \int_{-1/2}^{1/2} \cos 4\pi u \cdot \frac{2}{(4\pi)^2}$$

$$= \frac{1}{12} - \frac{1}{(4\pi)^2}\, 2 + 0$$

$$= \frac{1}{12} - \frac{1}{8\pi^2}$$