1 Any analytic function

We can expand around x_0

$$f(x) = f(x_0) + f'(x_0)\delta x + f''(x_0)\frac{(\delta x^2)}{2!} + \dots$$

where $\delta x = x - x_0$

2 Trig and Exponential

$$\begin{aligned}
\sin(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} ... \\
\cos(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} ... \\
e^x &= 1 + x + \frac{x^2}{2!} + ...
\end{aligned}$$

Show that $e^{ix} = \cos(x) + i\sin(x)$ by comparing power series

3 Power and Log

$$\begin{aligned}
\frac{1}{1+x} &= 1 - x + x^2 + ... \\
\log(1+x) &= x - \frac{x^2}{2} + \frac{x^3}{3} + ... \\
(1+x)^\alpha &= 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + ...
\end{aligned}$$

Show that $\int^x dx' \frac{1}{1+x'} = \log(1+x)$ by integrating term by term

4 Examples from quizzes

1. For $\beta \ll 1$:
$$\sqrt{\frac{1-\beta}{1+\beta}} \approx 1 - \beta + O(\beta^2)$$

2. For $p \ll T$
$$\frac{1}{e^{p/T} - 1} \approx \frac{T}{p} + O(1)$$

3. For $p \gg T$, $e^{p/T} \gg T$ and $e^{-p/T} \ll 1$
$$\frac{1}{e^{\frac{p}{T}} - 1} = \frac{e^{-p/T}}{1 - e^{-p/T}} \approx e^{-p/T} + (e^{-p/T})^2 + O((e^{-p/T})^3)$$

4. For $p \gg T$, $e^{p/T} \gg T$ and $e^{-p/T} \ll 1$
$$\frac{1}{e^{\frac{p}{T}} + 1} = \frac{e^{-p/T}}{1 + e^{-p/T}} \approx e^{-p/T} - (e^{-p/T})^2 + O((e^{-p/T})^3)$$

5. For $p \ll T$
$$\frac{1}{e^{\frac{p}{T}} + 1} \approx \frac{1}{2} - \frac{p}{4T} + O((p/T)^2)$$