Consider a hollow charged conducting sphere with charge Q and radius R. It is well known that the potential is

$$\Phi(r) = \frac{Q}{R}\theta(R, r) + \frac{Q}{r}\theta(r, R) \tag{0.1}$$

Let's use this to understand the Sturm-Louivillle problem. The homogeneous solutions to equations of Sturm-Louiville form satisfy

$$-\frac{d}{dx}p(x)\frac{d}{dx}y(x) + q(x)y(x) = 0\,, \tag{0.2}$$

while the Green function of this differential equation, satisfies the

$$-\frac{d}{dx}p(x)\frac{d}{dx}G(x, x') + q(x)G(x, x') = \delta(x - x') \tag{0.3}$$

1. Given two homogeneous solutions $y_1(x)$ and $y_2(x)$, show that the [Wronsk$(x)p(x)$] is constant, where Wronsk is the Wronskian (of course!), *i.e.* The Wronsk(x) = $y_1(x)y_2'(x) - y_2(x)y_1'(x)$.

2. Show that the Green function of the Sturm Louiville problem is

$$G(x, x') = \text{const}\ [y_1(x)y_2(x')\theta(x, x') + y_1(x')y_2(x)\theta(x', x)] \tag{0.4}$$

   and find the constant in terms of the Wronskian. The $y_1(x)$ and $y_2(x)$ must satisfy the appropriate boundary conditions as $x \to \infty$ and $x \to 0$ (if that is the domain of interest). Sometimes $y_1(x)$ is called $y_>(x)$ and $y_2(x)$ is called $y_<(x)$ – Why? Also, the understandable equation eq. (0.4) is often written in a confusing way:

$$G(x, x') = \text{const}\,[y_1(x_>)y_2(x_<)]\ , \tag{0.5}$$

   where $x_>$ ($x_<$) is the greater (lesser) of x and x'. I much prefer eq. (0.4)), since I know how to differentiate a θ-fcn.

3. Starting with the Poisson equation, show that the potential due to the charged sphere satisfies $\Phi(r)$

$$-\frac{1}{r^2}\frac{d}{dr}r^2\frac{d}{dr}\Phi = \frac{Q}{r^2}\delta(r - R)\,. \tag{0.6}$$

   (Most (all?) equations of linearized-physics are of Sturm-Louiville type) The two solutions to this equation are $1/r$ and const.

4. Use the Sturm-Louiville theory we developed to write down the Green function for a charged sphere. What solution do you take for y_1 and why. Why not a linear combination?

5. What is the physical interpretation of the fact that Wronskian is constant? This is generically the interpretation of this mathematical fact.