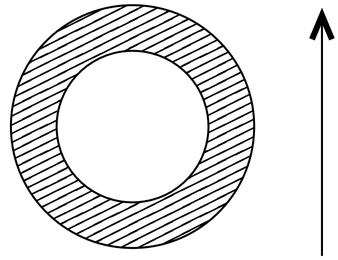
Problem 1. Electric field in a dielectric shell.

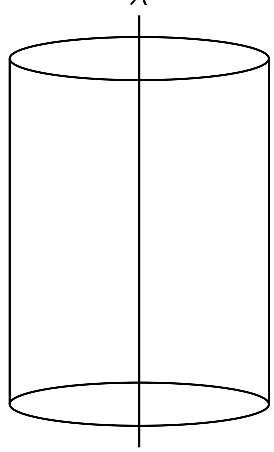


A spherical dielectric shell, with a hollow interior, has inner radius a and outer radius b and dielectric constant ε. The dielectric shell sits in an external electric field of magnitude E_o pointing in the z direction.

(a) Find the system equations which determines the electric field inside the sphere, but (for lack of time) do not try to solve this system.

(b) Taking $a \to 0$ (so that the sphere is solid) determine the electric field within the sphere for $r < b$.

Problem 2. Induced rotation.



An infinitely long dielectric wire runs along the z axis with charge density λ, and is surrounded by a thin dielectric cylindrical shell with radius a, carrying charge density $\sigma = -\lambda/(2\pi a)$. The suspended cylinder can rotate freely about the z-axis, but is initially at rest in a constant magnetic field in the z-direction, $\boldsymbol{B}_{\rm ext} = B_o\hat{\boldsymbol{z}}$

(a) Determine the electric field for $t < 0$.

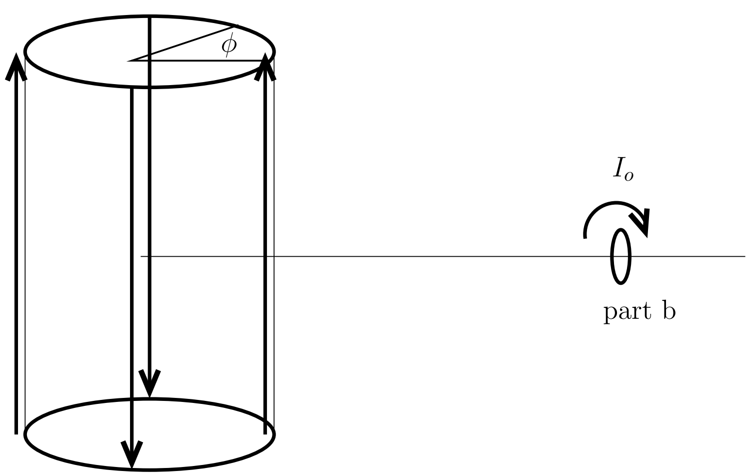
(b) At $t = 0$ we slowly reduce the magnetic field to zero over a time $T \gg a/c$. What happens and why? Draw a sketch of the resulting motion indicating the way that the cylinder rotates.

(c) Find the angular velocity of the cylinder as a function of time, taking its moment of inertia per unit length to be I.

(d) How did the condition $T \gg a/c$ help you in part (c) to find an approximate solution to the Maxwell equations. Point to a specific term in the Maxwell equations which was neglected/dropped/approximated using this condition. Give an estimate for the magnitude of the corrections to your result.

(e) Calculate the angular momentum per unit length for $t > T$ and show that it is conserved, *i.e.* that the final angular momentum equals the angular momentum for $t < 0$.

Problem 3. Currents in a cylindrical shell.



An infinite cylindrical shell of radius, a, carries a surface current in the z direction which is a function of angle, $\boldsymbol{K}(\phi) = K_o \cos(2\phi)\hat{\boldsymbol{z}}$.

(a) Determine the magnetic field outside and inside the shell produced by the surface current.

(b) A second *small* circular loop lies carries current I_o and has radius r_o and sits on the x-axis at distance ρ_o from the center. The current is oriented as shown. Determine the magnitude of the force on the current loop as a function of distance ρ from the center of the cylinder. What is the direction of the force.

Problem 4. Reflection and transmission from a plane of glass

Consider a plane wave of light in vacuum normally incident (*i.e.* head on) on a semi-infinite slab of glass filling the space $z > 0$. The glass has index of refraction $n > 1$ and magnetic permeability $\mu \simeq 1$. The frequency of the light is ω.

(a) Starting from the Maxwell equations, determine the amplitudes of the reflected and transmitted waves.

(b) Determine the reflection and transmission coefficients, *i.e.* the ratio of the reflected and transmitted power to the incident power.

(c) Determine the time averaged electromagnetic stress tensor on both sides of the interface, and use this to determine the force per unit area on the front face of the glass.

Grad, Div, Curl, and Laplacian

CARTESIAN $\quad d\boldsymbol{\ell} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}} \qquad d^3r = dx\,dy\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial x}\hat{\mathbf{x}} + \frac{\partial\psi}{\partial y}\hat{\mathbf{y}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right)\hat{\mathbf{x}} + \left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right)\hat{\mathbf{y}} + \left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} + \frac{\partial^2\psi}{\partial z^2}$$

CYLINDRICAL $\quad d\boldsymbol{\ell} = d\rho\,\hat{\boldsymbol{\rho}} + \rho\,d\phi\,\hat{\boldsymbol{\phi}} + dz\,\hat{\mathbf{z}} \qquad d^3r = \rho\,d\rho\,d\phi\,dz$

$$\nabla\psi = \frac{\partial\psi}{\partial\rho}\hat{\boldsymbol{\rho}} + \frac{1}{\rho}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}} + \frac{\partial\psi}{\partial z}\hat{\mathbf{z}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{\rho}\frac{\partial}{\partial\rho}(\rho A_\rho) + \frac{1}{\rho}\frac{\partial A_\phi}{\partial\phi} + \frac{\partial A_z}{\partial z}$$

$$\nabla\times\mathbf{A} = \left(\frac{1}{\rho}\frac{\partial A_z}{\partial\phi} - \frac{\partial A_\phi}{\partial z}\right)\hat{\boldsymbol{\rho}} + \left(\frac{\partial A_\rho}{\partial z} - \frac{\partial A_z}{\partial\rho}\right)\hat{\boldsymbol{\phi}} + \frac{1}{\rho}\left[\frac{\partial}{\partial\rho}(\rho A_\phi) - \frac{\partial A_\rho}{\partial\phi}\right]\hat{\mathbf{z}}$$

$$\nabla^2\psi = \frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial\psi}{\partial\rho}\right) + \frac{1}{\rho^2}\frac{\partial^2\psi}{\partial\phi^2} + \frac{\partial^2\psi}{\partial z^2}$$

SPHERICAL $\quad d\boldsymbol{\ell} = dr\,\hat{\mathbf{r}} + r\,d\theta\,\hat{\boldsymbol{\theta}} + r\sin\theta\,d\phi\,\hat{\boldsymbol{\phi}} \qquad d^3r = r^2\sin\theta\,dr\,d\theta\,d\phi$

$$\nabla\psi = \frac{\partial\psi}{\partial r}\hat{\mathbf{r}} + \frac{1}{r}\frac{\partial\psi}{\partial\theta}\hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\phi}\hat{\boldsymbol{\phi}}$$

$$\nabla\cdot\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta A_\theta) + \frac{1}{r\sin\theta}\frac{\partial A_\phi}{\partial\phi}$$

$$\nabla\times\mathbf{A} = \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta A_\phi) - \frac{\partial A_\theta}{\partial\phi}\right]\hat{\mathbf{r}} + \left[\frac{1}{r\sin\theta}\frac{\partial A_r}{\partial\phi} - \frac{1}{r}\frac{\partial}{\partial r}(rA_\phi)\right]\hat{\boldsymbol{\theta}} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right]\hat{\boldsymbol{\phi}}$$

$$\nabla^2\psi = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2}$$

Vector Identities

$$\mathbf{a}\cdot(\mathbf{b}\times\mathbf{c}) = \mathbf{b}\cdot(\mathbf{c}\times\mathbf{a}) = \mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})$$

$$\mathbf{a}\times(\mathbf{b}\times\mathbf{c}) = (\mathbf{a}\cdot\mathbf{c})\mathbf{b} - (\mathbf{a}\cdot\mathbf{b})\mathbf{c}$$

$$(\mathbf{a}\times\mathbf{b})\cdot(\mathbf{c}\times\mathbf{d}) = (\mathbf{a}\cdot\mathbf{c})(\mathbf{b}\cdot\mathbf{d}) - (\mathbf{a}\cdot\mathbf{d})(\mathbf{b}\cdot\mathbf{c})$$

$$\nabla\times\nabla\psi = 0$$

$$\nabla\cdot(\nabla\times\mathbf{a}) = 0$$

$$\nabla\times(\nabla\times\mathbf{a}) = \nabla(\nabla\cdot\mathbf{a}) - \nabla^2\mathbf{a}$$

$$\nabla\cdot(\psi\mathbf{a}) = \mathbf{a}\cdot\nabla\psi + \psi\nabla\cdot\mathbf{a}$$

$$\nabla\times(\psi\mathbf{a}) = \nabla\psi\times\mathbf{a} + \psi\nabla\times\mathbf{a}$$

$$\nabla(\mathbf{a}\cdot\mathbf{b}) = (\mathbf{a}\cdot\nabla)\mathbf{b} + (\mathbf{b}\cdot\nabla)\mathbf{a} + \mathbf{a}\times(\nabla\times\mathbf{b}) + \mathbf{b}\times(\nabla\times\mathbf{a})$$

$$\nabla\cdot(\mathbf{a}\times\mathbf{b}) = \mathbf{b}\cdot(\nabla\times\mathbf{a}) - \mathbf{a}\cdot(\nabla\times\mathbf{b})$$

$$\nabla\times(\mathbf{a}\times\mathbf{b}) = \mathbf{a}(\nabla\cdot\mathbf{b}) - \mathbf{b}(\nabla\cdot\mathbf{a}) + (\mathbf{b}\cdot\nabla)\mathbf{a} - (\mathbf{a}\cdot\nabla)\mathbf{b}$$

Integral Identities

$$\int_V d^3r\,\nabla\cdot\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\cdot\mathbf{A}$$

$$\int_V d^3r\,\nabla\psi = \int_S dS\,\hat{\mathbf{n}}\psi$$

$$\int_V d^3r\,\nabla\times\mathbf{A} = \int_S dS\,\hat{\mathbf{n}}\times\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\cdot\nabla\times\mathbf{A} = \oint_C d\boldsymbol{\ell}\cdot\mathbf{A}$$

$$\int_S dS\,\hat{\mathbf{n}}\times\nabla\psi = \oint_C d\boldsymbol{\ell}\,\psi$$

$\sqrt{4\pi/\mu_0}$. Therefore, (B.21) transforms to

$$d^3r\, \mathbf{r} \times \sqrt{4\pi\epsilon_0}\mathbf{j}_G, \qquad \text{(B.23)}$$

$$\int d^3r\, \mathbf{r} \times \mathbf{j}_G. \qquad \text{(B.24)}$$

C.1 Legendre Polynomials

The real-valued Legendre polynomials, $P_\ell(x)$, are defined by the generating function

$$\frac{1}{\sqrt{1-2xt+t^2}} \equiv \sum_{\ell=0}^{\infty} t^\ell P_\ell(x) \qquad |x| \le 1,\ 0 < t < 1. \qquad \text{(C.1)}$$

Alternatively, consider the differential equation

$$(1-x^2)\frac{d^2P(x)}{dx^2} - 2xy\frac{dP(x)}{dx} + \ell(\ell+1)P(x) = 0. \qquad \text{(C.2)}$$

The $P_\ell(x)$ are the eigenfunctions of the Sturm-Liouville eigenvalue problem defined by (C.2) on the interval $-1 \le x \le 1$ with the boundary conditions that $P(1)$ and $P(-1)$ are finite. The index ℓ is a non-negative integer. The polynomials are orthogonal,

$$\int_{-1}^{1} dx\, P_\ell(x)P_m(x) = \frac{2}{2\ell+1}\delta_{\ell m}, \qquad \text{(C.3)}$$

and complete,

$$\sum_{\ell=0}^{\infty}\left(\ell+\tfrac{1}{2}\right) P_\ell(x)P_\ell(x') = \delta(x-x'). \qquad \text{(C.4)}$$

Using the *Rodriguez formula*,

$$P_\ell(x) = \frac{1}{2^\ell \ell!}\frac{d^\ell}{dx^\ell}(x^2-1)^\ell, \qquad \text{(C.5)}$$

we find

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}(3x^2-1) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}(35x^4-30x^2+3). \end{aligned} \qquad \text{(C.6)}$$