1 Series of functions

In each case we are expanding a function in a complete set of eigen-functions

$$\langle x|F\rangle = \sum_n \langle x|n\rangle \langle n|F\rangle \tag{1}$$

We require that the fucntions are complete (in the space of functions which satisfy the same boundary conditions as F) and orthogonal

$$\sum_n |n\rangle \langle n| = I \qquad \langle n_1|n_2\rangle = \delta_{n_1 n_2} \tag{2}$$

In what follows we show the eigen-function in square brackets

(a) A 2π periodic function $F(\phi)$ is expandable

$$F(\phi) = \sum_{m=-\infty}^{\infty} \left[e^{im\phi}\right] F_m \tag{3}$$

$$F_m = \int_0^{2\pi} \frac{d\phi}{2\pi} \left[e^{-im\phi}\right] F(\phi) \tag{4}$$

$$\int_0^{2\pi} d\phi \left[e^{-im\phi}\right] \left[e^{im'\phi}\right] = 2\pi\delta_{mm'} \tag{5}$$

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \sum_n \delta(\phi - \phi' + 2\pi n) \tag{6}$$

(b) A square interable function in one dimension has a fourier transform

$$F(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{ikz}\right] F(k) \tag{7}$$

$$F(k) = \int_{-\infty}^{\infty} dz \left[e^{-ikz}\right] F(z) \tag{8}$$

$$\int_{-\infty}^{\infty} dz\, e^{-iz(k-k')} = 2\pi\delta(k - k') \tag{9}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} = \delta(z - z') \tag{10}$$

(c) A regular function on the sphere (θ, ϕ) can be expanded in spherical harmonics

$$F(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [Y_{\ell m}(\theta, \phi)]\, F_{\ell m} \tag{11}$$

$$F_{\ell m} = \int d\Omega\, [Y^*_{\ell m}(\theta, \phi)]\, F(\theta, \phi) \tag{12}$$

$$\int d\Omega\, [Y^*_{\ell m}(\theta, \phi)]\, [Y_{\ell' m'}(\theta, \phi)] = \delta_{\ell\ell'}\delta_{mm'} \tag{13}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [Y_{\ell m}(\theta, \phi)]\, [Y^*_{\ell m}(\theta', \phi')] = \delta(\cos\theta - \cos\theta')\delta(\phi - \phi') \tag{14}$$

(d) A function, $F(\rho)$ on the half line $\rho = [0, \infty]$, which vanishes like ρ^m as $\rho \to 0$ can be expanded in Bessel functions. This is known as a Hankel transform and arises in cylindrical coordinates

$$F(\rho) = \int_0^\infty kdk \; [J_m(k\rho)] \; F_m(k) \tag{15}$$

$$F_m(k) = \int_0^\infty \rho d\rho \; [J_m(k\rho)] \; F(\rho) \tag{16}$$

$$\int_0^\infty \rho d\rho \; [J_m(\rho k)] \, [J_m(\rho k')] = \frac{1}{k}\delta(k - k') \tag{17}$$

$$\int_0^\infty kdk \; [J_m(\rho k)] \, [J_m(\rho' k)] = \frac{1}{\rho}\delta(\rho - \rho') \tag{18}$$