Problem 1. Fourier Transforms

(a) Write down Maxwell equations in Fourier space, *i.e.* writing $\boldsymbol{E}$ and $\boldsymbol{B}$, ρ and $\mathbf{j}$ as Fourier transforms

$$\boldsymbol{E}(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} dt \int \mathrm{d}^3 r e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{E}(t, \boldsymbol{r}) \tag{1}$$

$$\boldsymbol{B}(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} dt \int \mathrm{d}^3 r e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} \boldsymbol{B}(t, \boldsymbol{r}) \tag{2}$$

$$\rho(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} dt \int \mathrm{d}^3 r e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} \rho(t, \boldsymbol{r}) \tag{3}$$

$$\mathbf{j}(\omega, \boldsymbol{k}) = \int_{-\infty}^{\infty} dt \int \mathrm{d}^3 r e^{i\omega t - i\boldsymbol{k}\cdot\boldsymbol{r}} \mathbf{j}(t, \boldsymbol{r}) \tag{4}$$

write down the equations of motion for $\boldsymbol{E}(\omega, \boldsymbol{k})$ and $\boldsymbol{B}(\omega, \boldsymbol{k})$.

(b) The screened Coulomb potential, known as the Yukawa potential is

$$V(\boldsymbol{r}) = \frac{e^{-m|\boldsymbol{r}|}}{4\pi|\boldsymbol{r}|}\,, \tag{5}$$

with $m > 0$. What is $V(\boldsymbol{k}) = \int \mathrm{d}^3 r\, e^{i\boldsymbol{k}\cdot\boldsymbol{r}} V(\boldsymbol{r})$? What is the limit of $V(\boldsymbol{k})$ as $m \to 0$?

(c) What is the Poisson equation in Fourier space:

$$-\nabla^2 \varphi = \rho(\boldsymbol{x})\,. \tag{6}$$

(d) Use Fourier transforms to heuristically explain why if

$$\nabla \times \boldsymbol{E}(\boldsymbol{x}) = 0 \tag{7}$$

then $\boldsymbol{E}$ can be written as the gradient of a scalar function $\boldsymbol{E} = -\nabla\varphi$

Problem 2. Zangwill 1.4: Vector Derivative Identities (optional)

Problem 3. An non-uniformly charged spherical shell

A hollow spherical shell of radius R is made of insulating material, and has a charge per unit area:

$$\sigma(\theta, \phi) = \sigma_o \left(\cos\theta + \tfrac{1}{2}\sin\theta\cos\phi\right) \tag{8}$$

(a) Find the potential for $r < R$ and $r > R$.

(b) From the asymptotics of your solution, determine the dipole moment $\boldsymbol{p}$ in Cartesian coordinates $\boldsymbol{p} = p_x\hat{\mathbf{x}} + p_y\hat{\mathbf{y}} + p_z\hat{\mathbf{z}}$.

(c) Determine the electric field inside the sphere in Cartesian coordinates.

Problem 4. Green function of a sphere

Consider a grounded, metallic, hollow spherical shell of radius R. A point charge of charge q is placed at a distance, a, from the center of the sphere along the x-axis. For simplicity take $a > R$.

(a) Determine the potential $\varphi(\boldsymbol{x})$. (Hint: consider an image charge at radius R^2/a).

(b) Determine the total charge on the sphere.

(c) Now consider a point charge of charge q at a distance z above a metallic hemisphere of radius R in contact with a grounded plane. Determine the force on the charge as a function of z.

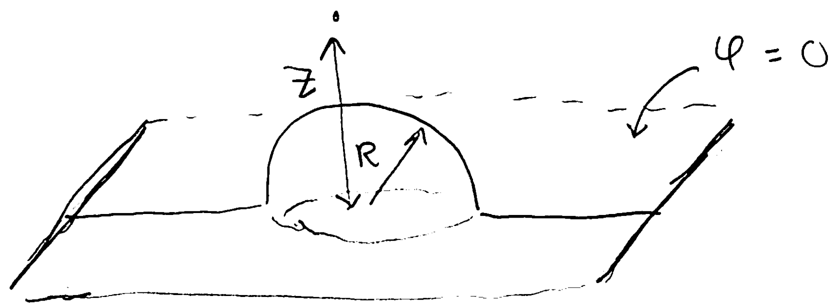

Figure: Hemisphere with a plane and a point charge at height z

Problem 5. Defects

This problem will study defects in parallel plate capacitors. A parallel plate capacitor has area, A, and separation, D, and is maintained at the potential difference, $\Delta V = E_o D$. There are n defects per unit area on the lower plate and none on the upper. The defects consist of hemispherical shells of radius a bending towards the upper plate. You should assume that $a \ll D$, and that $na^2 \ll 1$ so that the defects are very widely spaced.

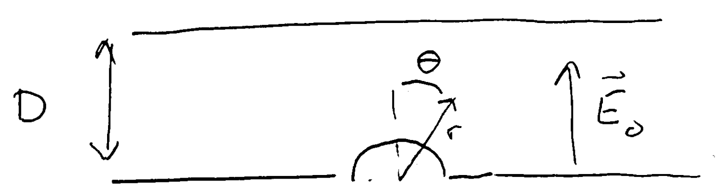

Figure: A defect on a capacitor plate.

(a) Determine the charge per unit area on and near the defect. Plot the surface charge on the hemisphere as a function of θ, and on the plane as a function of r. (Hint: To solve for the potential in the vicinity of a defect use that fact that for $a \ll z \ll D$ the potential reaches its unperturbed form $\varphi(z) = -E_o z$, so that the upper boundary can be ignored.)

(b) Show that the charged induced on the hemisphere is:

$$Q = E_o a^2\, 3\pi \tag{9}$$

(c) Use these results to show that the capacitance is unchanged by the defect to the order we are working, *i.e.*

$$C \simeq \frac{A}{d} \tag{10}$$

(d) In deriving this result we have used that $D \gg a$. The size of corrections to the potential you found are of order $\sim a^3/D^3$. Explain why.

Problem 6. Problem 7.19: A Periodic Array of Charged Rings.

In addition:

(a) Briefly explain why

$$I'_\alpha(y) K_\alpha(y) - I_\alpha(y) K'_\alpha(y) = \frac{1}{y} \tag{11}$$

(b) Determine the first correction to the asymptotic form of the potential far from the rings, $\rho \gg R$.

Problem 7. Practice with the stress tensor:

(a) Within the limits of electrostatics, show that the electric force on a charged body is related to a surface integral of the (electric) stress tensor:

$$F^j = \int_V d^3\boldsymbol{r}\, \rho(\boldsymbol{x})\, E^j = -\int_S dS\, n_i T_E^{ij} \tag{12}$$

where $T_E^{ij} = -E^i E^j + \frac{1}{2} E^2 \delta^{ij}$

(b) Use this result to calculate the force between two (solid and insulating) uniformly charged hemispheres each with total charge Q and radius R that are separated by a small gap as shown below.

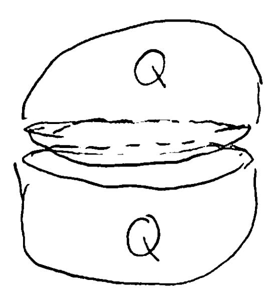

Figure: Two hemispheres