Problem 1. Energy during a burst of deceleration

A particle of charge e moves at constant velocity, βc, for $t < 0$. During the short time interval, $0 < t < \Delta t$ its velocity remains in the same direction but its speed decreases linearly in time to zero. For $t > \Delta t$, the particle remains at rest.

(a) Show that the radiant energy emitted per unit solid angle is

$$\frac{dW}{d\Omega} = \frac{e^2\beta^2}{64\pi^2 c\Delta t} \frac{(2-\beta\cos\theta)\left[1+(1-\beta\cos\theta)^2\right]\sin^2\theta}{(1-\beta\cos\theta)^4} \tag{1}$$

(b) In the limit $\gamma \gg 1$, show that the angular distribution can be expressed as

$$\frac{dW}{d\xi} \simeq \frac{e^2\beta^2}{4\pi\, c} \frac{\gamma^4}{\Delta t} \frac{\xi}{(1+\xi)^4} \tag{2}$$

where $\xi = (\gamma\theta)^2$.

(c) Show for $\gamma \gg 1$ that the total energy radiated is in agreement with the relativistic generalization of the Larmour formula.

Problem 2. An oscillator radiating

(a) Determine the time averaged power radiated per unit sold angle for a *non-relativistic charge* moving along the z-axis with instantaneous position, $z(T) = H\cos(\omega_o T)$.

(b) Now consider relativistic charge executing simple harmonic motion. Show that the instantaneous power radiated per unit solid angle is

$$\frac{dP(T)}{d\Omega} = \frac{dW}{dT\, d\Omega} = \frac{e^2}{16\pi^2} \frac{c\beta^4}{H^2} \frac{\sin^2\theta\cos^2(\omega_o T)}{(1+\beta\cos\Theta\sin\omega_o T)^5} \tag{3}$$

Here $\beta = \omega_o H/c$ and $\gamma = 1/\sqrt{1-\beta^2}$

(c) In the relativistic limit the power radiated is dominated by the energy radiated during a short time interval around $\omega_o T = \pi/2,\, 3\pi/2,\, 5\pi/2, \ldots$. Explain why. Where does the outgoing radiation point at these times.

(d) Let ΔT denote the time deviation from one of this discrete times, *e.g.* $T = 3\pi/(2\omega_o) + \Delta T$. Show that close to one of these time moments:

$$\frac{dP(\Delta T)}{d\Omega} = \frac{dW}{d\Delta T\, d\Omega} \simeq \frac{2e^2}{\pi^2} \frac{c\beta^4}{H^2} \gamma^6 \frac{(\gamma\omega_o\Delta T)^2(\gamma\theta)^2}{(1+(\gamma\theta)^2+(\gamma\omega_o\Delta T)^2)^5} \tag{4}$$

(e) By integrating the results of the previous part over the ΔT of a single pulse, show that the time averaged power is

$$\overline{\frac{dP(T)}{d\Omega}} = \frac{e^2}{128\pi^2} \frac{c\beta^4}{H^2} \gamma^5 \frac{5(\gamma\theta)^2}{(1+(\gamma\theta)^2)^{7/2}} \tag{5}$$

(f) Make rough sketches of the angular distribution for non-relativistic and relativistic motion.

Problem 3. Periodic pulses

Consider a periodic motion that repeats itself with period $\mathcal{T}_o$. Show that the continuous frequency spectrum becomes a discrete spectrum containing frequencies that are integral multiples of the fundamental, $\omega_o = 2\pi/\mathcal{T}_o$.

Let the electric field from a single pulse (or period) be $E_1(t)$, *i.e.* where $E_1(t)$ is non-zero between 0 and $\mathcal{T}_o$ and vanishes elsewhere, $t < 0$ and $t > \mathcal{T}_o$. Let $E_1(\omega)$ be its fourier transform.

(a) Suppose that the wave form repeats once so that two pulses are received. $E_2(t)$ consists of the first pulse $E_1(t)$, plus a second pulse, $E_2(t) = E_1(t) + E_1(t - \mathcal{T}_o)$. Show that the Fourier transform and the power spectrum is

$$E_2(\omega) = E_1(\omega)\,(1 + e^{i\omega\mathcal{T}_o}) \qquad |E_2(\omega)|^2 = |E_1(\omega)|^2\,(2 + 2\cos(\omega\mathcal{T}_o)) \tag{6}$$

(b) Now suppose that we have n (with n odd) arranged almost symmetrically around $t = 0$, *i.e.*

$$E_n(t) = E_1(t+(n-1)\mathcal{T}_o/2) + \ldots + E_1(t+\mathcal{T}_o) + E_1(t) + E_1(t-\mathcal{T}_o) + \ldots E_1(t-(n-1)\mathcal{T}_o/2)\,, \tag{7}$$

so that for $n = 3$

$$E_3(t) = E_1(t + \mathcal{T}_o) + E_1(t) + E_1(t - \mathcal{T}_o)\,. \tag{8}$$

Show that

$$E_n(\omega) = E_1(\omega)\,\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)} \tag{9}$$

and

$$|E_n(\omega)|^2 = |E_1(\omega)|^2\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\sin(\omega\mathcal{T}_o/2)}\right)^2 \tag{10}$$

(c) By taking limits of your expressions in the previous part show that after n pulses, with $n \to \infty$, we find

$$E_n(\omega) = \sum_m E_1(\omega_m)\frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m) \tag{11}$$

and

$$|E_n(\omega)|^2 = \underbrace{n\mathcal{T}_o}_{\text{total time}} \times \sum_m |E_1(\omega_m)|^2\frac{2\pi}{\mathcal{T}_o^2}\delta(\omega - \omega_m) \tag{12}$$

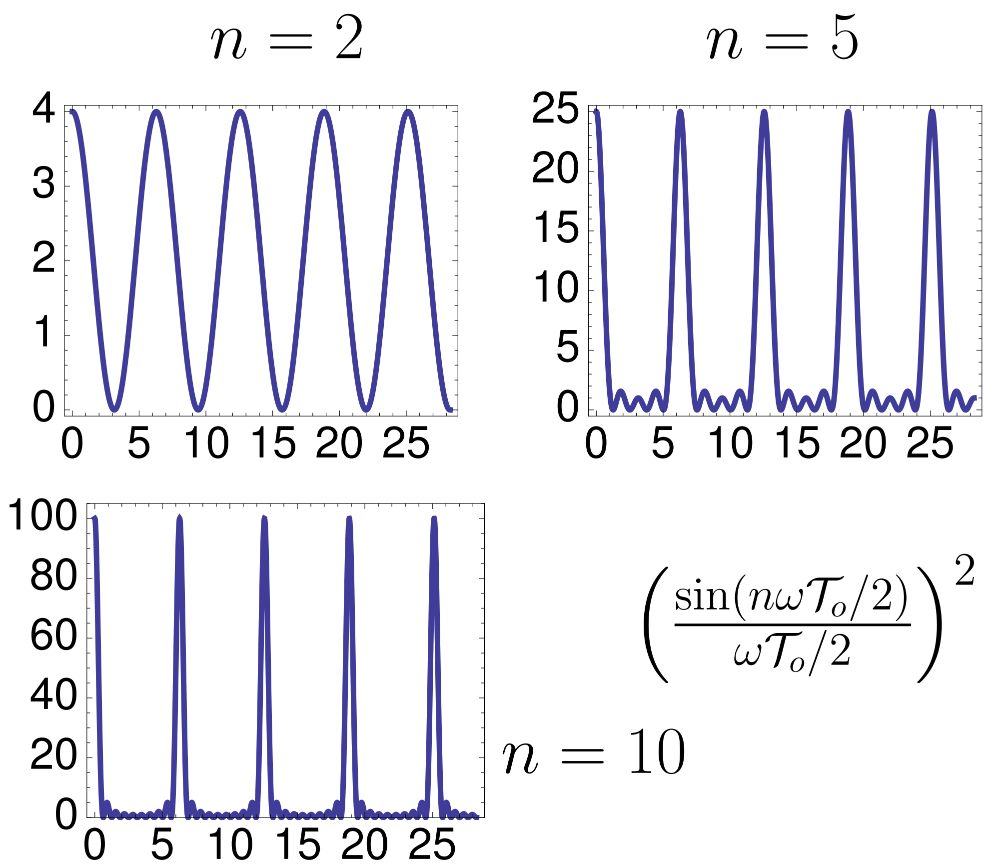
where $\omega_m = 2\pi m/\mathcal{T}_o$.

Remark We have in effect shown that if we define

$$\Delta(t) \equiv \sum_{n=-\infty}^{\infty} \delta(t - n\mathcal{T}_o)\,. \tag{13}$$

Then the Fourier transform of $\Delta(t)$ is

$$\hat{\Delta}(\omega) = \sum_n e^{-i\omega n\mathcal{T}_o} = \sum_m \frac{2\pi}{\mathcal{T}_o}\delta(\omega - \omega_m)\,. \tag{14}$$



$n=2$
4 3 2 1 0
0 5 10 15 20 25
$n=5$
25 20 15 10 5 0
0 5 10 15 20 25
100 80 60 40 20 0
0 5 10 15 20 25
$n=10$
$\left(\frac{\sin(n\omega\mathcal{T}_o/2)}{\omega\mathcal{T}_o/2}\right)^2$

(d) Show that a general expression for the time averaged power radiated per unit solid angle into each multipole $\omega_m \equiv m\omega_o$ is:

$$\frac{dP_m}{d\Omega} = \frac{|rE(\omega_m)|^2}{\mathcal{T}_o^2} \tag{15}$$

Or

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2\omega_o^4 m^2}{32\pi^4 c^3} \left| \int_0^{\mathcal{T}_o} \mathbf{v}(T) \times \boldsymbol{n} \exp\left[i\omega_m (T - \frac{\boldsymbol{n}\cdot r_*(T)}{c}) \right] \right|^2 \mathrm{d}T\,, \tag{16}$$

Here $d\hat{P}_m/d\Omega$ is defined so that over along time period $\Delta\mathcal{T}$, the energy per solid angle is

$$\frac{dW}{d\Omega} = \Delta\mathcal{T} \sum_{m=1}^{\infty} \frac{d\hat{P}_m}{d\Omega} \tag{17}$$

Also note that we are summing only over the positive values of m which is different from how we had it in class:

$$\frac{d\hat{P}_m}{d\Omega} \equiv \frac{dP_m}{d\Omega} + \frac{dP_{-m}}{d\Omega} \tag{18}$$

Problem 4. Radiation spectrum of a SHO

(a) Show that for the simple harmonic motion of a charge discussed in Problem 2 the average power radiated per unit solid angle in the m-th harmonic is

$$\frac{d\hat{P}_m}{d\Omega} = \frac{e^2 c\beta^2}{8\pi^2 H^2} m^2 \tan^2\theta \left[J_m(m\beta\cos\theta)\right]^2 \tag{19}$$

(b) Show that in the non-relativistic limit the total power radiated is all in the fundamental and has the value

$$P = \frac{e^2}{4\pi}\frac{2}{3}\omega_o^4 \overline{H^2} \tag{20}$$

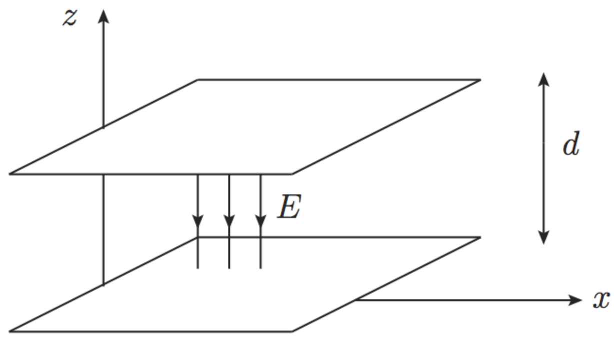
where $\overline{H^2}$ is the mean squared amplitude of the oscillation.

Problem 5. Physics of the relativistic stress tensor

Consider a capacitor at rest. The area of each plate is A, and the electric field between the plates is E. The plates are orthogonal to the $x-$axis (see figure). The rest mass of each plate is $M_{\rm pl}$. The plates are kept a distance d apart by four thin columns (not shown). We assume that each of these columns have mass $M_{\rm col}$, and there is a stress tensor in the columns due to the electric attraction of the plates. (There is also a surface stress tensor in the plates due to the electric repulsion of the charges on the plates, but you won't need this.)

(a) Write down the expression for the energy-momentum tensor of the electromagnetic field $\Theta^{\mu\nu}_{\rm em}$ in terms of the Maxwell field strength $F^{\mu\nu}$. Show that the total rest mass $Mc^2 = \int \mathrm{d}^3r\, \Theta^{00}_{\rm tot}$ of the capacitor setup is:

$$M_{\rm tot}c^2 = 2M_{\rm pl}c^2 + 4M_{\rm col}c^2 + \frac{1}{2}E^2 Ad \tag{21}$$



Remark. In practice the field term is very small compared to the first two terms, but we will include its effect in this problem.

(b) Determine the non-vanishing components of the electromagnetic stress tensor integrated over space:

$$\int \mathrm{d}^3 r\, \Theta_{\rm em}^{\alpha\beta}. \tag{22}$$

(Hints: $\int \Theta_{\rm em}^{xx}$, $\int \Theta_{\rm em}^{yy}$, $\int \Theta_{\rm em}^{zz}$, $\int \Theta_{\rm em}^{00}$ are non-zero.)

(c) Show that for a stationary configuration that

$$\int \mathrm{d}^3 r\, \Theta_{\rm tot}^{ij}(\boldsymbol{r}) = 0 \tag{23}$$

(Hints: Explain why $\partial_k \Theta_{\rm tot}^{kj} = 0$, and then study the expression $\partial_k(x^i \Theta_{\rm tot}^{kj})$)

(d) Determine $\int_{\rm col} \Theta_{\rm mech}^{zz}$ in the columns, and interpret your result physically by showing the forces involved with a free body diagram.

(e) Consider now an observer in frame K who is moving in the positive $z-$direction with velocity v relative to the rest frame of the capacitor. According to special relativity the energy of the capacitor in frame K is $\gamma M c^2$ where $\gamma = (1-(v/c)^2)^{1/2}$.

(i) Show that the integrated electromagnetic stress tensor in frame K, $\underline{\Theta}_{\rm em}^{00}$, is

$$\int d^3\underline{r}\, \underline{\Theta}_{\rm em}^{00}(\underline{r}) = \frac{1}{2} E^2 A d\, \sqrt{1-(v/c)^2} \tag{24}$$

Here $\underline{r}$ are the boosted coordinates.

(ii) Show that the integrated mechanical stress tensor including the plates and the columns

$$\int d^3\underline{r}\, \underline{\Theta}_{\rm mech}^{00}(\underline{r}) = \gamma\,(2M_{\rm pl}c^2 + 4M_{\rm col}c^2) + \frac{1}{2}E^2 A d\, \frac{(v/c)^2}{\sqrt{1-(v/c)^2}} \tag{25}$$

(iii) Use these results to compute

$$\int d^3\underline{r}\, \underline{\Theta}_{\rm tot}^{00}(\underline{r}) \tag{26}$$

in frame K and comment on the result.