Problem 1. A cylinder in a magnetic field

A very long hollow cylinder of inner radius a and outer radius b of permeability μ is placed in an initially uniform magnetic field $\boldsymbol{B}_o$ at right angles to the field.

(a) For a constant field B_o in the x direction show that $A^z = -B_o y$ is the vector potential. This should give you an idea of a convenient set of coordinates to use.

Remark: See Wikipedia for a list of the vector Laplacian in all coordinates. Most often the vector Laplacian is used if the current is azimuthal and solutions may be looked for with $A_\phi \neq 0$ and $A_r = A_\theta = 0$ (or $A_\rho = A_z = 0$ in cylindrical coordinates). This could be used for example in Problem 3. Similarly if the current runs up and down, with $A_z \neq 0$ and $A_\rho = A_\phi = 0$, so that $\boldsymbol{B} = (B_x(x,y,z), B_y(x,y,z), 0)$ is independent of z, then the vector Laplacian in cylindrical coordinates $-\nabla^2 A_z$ is a good way to go.

(b) Show that the magnetic field in the cylinder is constant $\rho < a$ and determine its magnitude.

(c) Sketch $|\boldsymbol{B}|/|\boldsymbol{B}_o|$ at the center of the as function of μ for $a^2/b^2 = 0.9, 0.5, 0.1$.

Problem 2. Helmholtz coils

Consider a compact circular coil of radius a carrying current I, which lies in the $x - y$ plane with its center at the origin.

(a) By elementary means compute the magnetic field along the z axis.

(b) Show by direct analysis of the Maxwell equations $\nabla \cdot \boldsymbol{B} = 0$ and $\nabla \times \boldsymbol{B} = 0$ that slightly off axis near $z = 0$ the magnetic field takes the form

$$B_z \simeq \sigma_0 + \sigma_2 \left(z^2 - \tfrac{1}{2}\rho^2\right), \quad B_\rho \simeq -\sigma_2 z \rho \,, \tag{1}$$

where $\sigma_0 = (B_z^o)$ and $\sigma_2 = \frac{1}{2}\left(\frac{\partial^2 B_z^o}{\partial z^2}\right)$ are the field and its z derivatives evaluated at the origin. For later use give σ_0 and σ_2 explictly in terms of the current and the radius of the loop.

Remark: Upon solving this problem, it should be clear that this method of solution does not rely on being close to $z = 0$. We just chose $z = 0$ for definiteness.

(c) Now consider a second identical coil (co-axial with the first), having the same magnitude and direction of the current, at a height b above the first coil, where a is the radii of the rings. With the coordinate origin relocated at the point midway between the two centers of the coils, determine the magnetic field on the z-axis near the origin as an expansion in powers of z to z^4. Use mathematica if you like. You should find that the coefficient of z^2 vanishes when $b = a$

Remark For $b = a$ the coils are known as Helmholtz coils. For this choice of b the z^2 terms in part (c) are absent. (Also if the off-axis fields are computed along the lines of part (b), they also vanish.) The field near the origin is then constant to 0.1% for $z < 0.17\, a$.

Problem 3. The field from a shell of current.

Consider conducting ring of current radius a lying in the $x - y$ plane, carrying current I in the counter clockwise direction, $\boldsymbol{I} = I\hat{\boldsymbol{\phi}}$.

(a) Starting from the general (coulomb gauge) expression

$$\boldsymbol{A}(\boldsymbol{r}) = \int d^3\boldsymbol{r}_o \, \frac{\mathbf{j}(\boldsymbol{r}_o)/c}{4\pi|\boldsymbol{r} - \boldsymbol{r}_o|} \tag{2}$$

and the expansion of $1/(4\pi|\boldsymbol{r} - \boldsymbol{r}_o|)$ in spherical coordinates, show that the expansion of A_ϕ in the x, y plane inside the ring is

$$A_\phi(\rho)\big|_{z=0} = \frac{I}{2c}\sum_{\ell=1}^{\infty} \frac{(P_\ell^1(0))^2}{\ell(\ell+1)} \left(\frac{\rho}{a}\right)^\ell \tag{3}$$

where $\rho = \sqrt{x^2 + y^2}$ and P_ℓ^1 is the associated Legendre polynomial. (Check out wikipedia entry on spherical harmonics)

(b) Compute $B_z(\rho)$ in the x, y plane.

(c) Show that close to the axis of the shell the magnetic field you computed in part (b) is in agreement with the results of Eq. (1) when evaluated at $z = 0$.

Remark: Using the generating function of Legendre polynomials derived in class

$$\frac{1}{\sqrt{1 + r^2 - 2r\cos\theta}} = \sum_{\ell=0}^{\infty} r^\ell P_\ell(\cos\theta) \tag{4}$$

and the defintion of $P_\ell^1(\cos\theta) = -\sin\theta \frac{dP_\ell(\cos\theta)}{d(\cos\theta)}$, we show that

$$\sum_{\ell=1}^{\infty} r^\ell P_\ell^1(0) = \frac{-r}{(1+r^2)^{3/2}} \simeq -r + \frac{3}{2}r^3 - \frac{15}{8}r^5 + \ldots \tag{5}$$

establishing that

$$P_1^1(0) = -1 \qquad P_3^1(0) = \frac{3}{2} \qquad P_5^1(0) = -\frac{15}{8} \qquad P_\ell^1(0) = 0 \text{ for } \ell \text{ even.} \tag{6}$$

(d) Consider a magnetic dipole of magnetic moment $\boldsymbol{m} = -m\hat{\boldsymbol{z}}$ in the $x - y$ plane oriented oppositely to the field from the ring, show that the force on the dipole is

$$\mathbf{F} = -\hat{\boldsymbol{\rho}}\frac{mB_o}{a}\sum_{\ell=3}^{\infty} \frac{(\ell-1)}{\ell}(P_\ell^1(0))^2 \left(\frac{\rho}{a}\right)^{\ell-2} \tag{7}$$

where the negative indicates that the force is towards the center, and $B_o = I/(2ca)$ is the magnetic field in the center of the ring.

(e) (**Optional**) Plot the force $|\mathbf{F}|\,/\,[mB_o/a]$ as a function of ρ/a.

Problem 4. Two electrodes in a conductor filling half of space

Two small spherical electrodes of radius a are embedded in a semi-infinite medium of conductivity σ, each at a distance $d > a$ from the plane face of the medium and at a distance b from each other. Find the resistance between the electrodes. Sketch the flow lines of current if the two electrodes are held at a potential difference $\Delta\varphi$.

Problem 5. Magnetic field from a hard ferromagnet

A solid right cylindrical hard ferro-magnet of radius a and height h has a constant uniform magnetization $\boldsymbol{M} = M\hat{\boldsymbol{z}}$ directed along the axis of the cylinder. For definiteness place one end of the cylinder at $z = 0$ and the other end at $z = -h$.

This problem will determine the magnetic field $\boldsymbol{B}$ outside the magnet using the magnetic scalar potential.

(a) For certain problems involving currents confined to surfaces (this is the case here) the scalar magnetic potential can be a useful device. Take the maxwell equations

$$\nabla \times \boldsymbol{H} = \mathbf{j}/c \tag{8}$$

$$\nabla \cdot \boldsymbol{B} = 0 \tag{9}$$

In any current free region, since the curl is zero, we may write $\boldsymbol{H}$ as the gradient of a magnetic potential

$$\boldsymbol{H} = -\nabla\psi_m(\boldsymbol{r}) \tag{10}$$

and solve for ψ_m. The potential ψ_m can be matched across the interfaces using the boundary conditions

$$\boldsymbol{n} \times (\boldsymbol{H}_2 - \boldsymbol{H}_1) = \boldsymbol{K}/c \tag{11}$$

$$\boldsymbol{n} \cdot (\boldsymbol{B}_2 - \boldsymbol{B}_1) = 0 \tag{12}$$

For the problem at hand, show that both inside and outside the cylinder the magnetic potential is

$$\psi_m(\boldsymbol{r}) = \int d^3\boldsymbol{x} \frac{-\nabla \cdot \boldsymbol{M}}{4\pi|\boldsymbol{r} - \boldsymbol{x}|} \tag{13}$$

(b) Use the scalar potential to determine the magnetic field on the axis (but outside the cylinder) for $z > 0$. The solution was presented in class using the vector rather than the scalar potential, and you may wish to compare the integration regions in the two cases.

(c) For a long cylinder $h \gg a$ show that for $a \ll z \ll h$ that the magnetic potential and thus the magnetic field outside of the cylinder are described by the magnetic field of a magnetic monopole. Determine the magnetic charge.

Remark: the long thin cylinder in part (c) is often called the Dirac string.