Problem 1. Drude Model of Metals

The Drude model describes the interactions of n electrons per volume with the electric field by the drag model

$$m\frac{dv}{dt} + \frac{mv}{\tau_c} = eE(t) \tag{1}$$

We estimated previously that plasma frequency of the metal is $\omega_p \equiv \sqrt{ne^2/m} \sim 10^{15}\,\mathrm{s}^{-1}$. The time between collisions with impurities is of order $\tau_c \sim 10^{-13}\,\mathrm{s}$, and thus the dimensionless parameter $\omega_p\tau_c \sim 100$. We previously found the DC conductivity:

$$\sigma = \omega_p^2\tau_c \sim 10^{17}\,\mathrm{s}^{-1} \tag{2}$$

(a) Show that the constituent relation for the conductivity in the Drude model is

$$\sigma(\omega) = \frac{\sigma_o}{1 - i\omega\tau_c} \tag{3}$$

(b) Determine the real and imaginary parts of the $\epsilon(\omega)$. Sketch the real and imaginary parts of $\epsilon(\omega)$ as a function of ω/ω_p for large frequencies. Also sketch the real and imaginary parts at moderate frequency as a function $\omega\tau_c$. Be sure that you draw your curves approximately to scale, indicating where is one on both the x and y axes.

(c) Describe what the low and very high frequency behavior of these functions implies for the propagation of light in the metal. What is "very high frequency" for a metal like copper?

(d) Show that charge relaxation is governed by the equation

$$\partial_t\rho + \int_{-\infty}^{\infty} \sigma(t - t')\rho(t')dt' = 0 \tag{4}$$

(e) Show that $\sigma(t - t_o)$ is determined up to a constant by the retarded Green function of the differential equation, $G_R(t - t_o)$:

$$\left[m\frac{d}{dt} + \frac{m}{\tau_c}\right] G_R(t - t_o) = \delta(t - t_o) \tag{5}$$

(f) By direct integration in time of Eq. (5) show that

$$\sigma(\tau) = \theta(\tau)\omega_p^2 e^{-\tau/\tau_c} \tag{6}$$

Estimate the time scale for charge relaxation in Hz.

(g) Finally, by taking the inverse Fourier transform of Eq. (3) show that you get the same result.

Problem 2. Zangwill 17.22: A Photonic Band Gap Material

Problem 3. Zangwill 18.14: Energy Flow in the Lorentz Model

Problem 4. Frequency comb

Examining PRL 99, 263902 (2007), I came across the following sentence:

> PRL **99,** 263902 (2007) PHYSICAL RE
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> We use a 1 GHz mode locked Ti:sapphire laser with a bandwidth of about 12 THz FWHM centered at 785 nm and an average output power of about 0.5 W as a light source [10]. The spectroscopy resonator consists of broad-

Each laser pulse has a total energy of $0.5\,\mathrm{W}/(1\mathrm{GHz})$. Treat the electric field at a fixed location as the temporal pulse

$$E(t) = \text{gaussian}(t) \times e^{-i\omega_o t} \tag{7}$$

The "bandwidth at FWHM" refers to the full frequency width $\Delta f = \Delta\omega/2\pi$ of the power spectrum $|E(\omega)|^2$ of a single pulse, when the function $|E(\omega)|^2$ has reached half of its maximum.

(a) If there was only a single pulse, determine the power spectrum $|E(\omega)|^2$. Give all parameters (such as the width and height) numerically.

Now consider a periodic sequence of pulses with a repetition rate of $f = 1\,\mathrm{GHz} \pm 1\,\mathrm{Hz}$, where the $\pm 1\,\mathrm{Hz}$ indicates the uncertainty of the GHz rep-rate.

$$\tag{8}$$

(b) Neglecting the uncertainty of the rep-rate, determine the power spectrum $|E(\omega)|^2$. Give all parameters numerically

(c) Draw several qualitative graphs of the power spectrum approximately to scale, which explain the meaning of your formula in part (b). Take into account (qualitatively) the uncertainty in the GHz clock. On each graph, be sure to give the units you are using on both the x and y axes.

Remark: Such frequency combs are remarkably useful, producing coherent light over a wide range of well defined frequencies. Frequency combs have been used by Tom Allison in our department.

Problem 5. Green theorem for first and second order equations and the initial value problem

First order: Consider a model first order equation equation for the velocity

$$m\frac{dv}{dt} + m\eta v = 0 \tag{9}$$

describing how a particle slows down.

(a) (**Optional**-already assigned) Determine the Green function for the equation *i.e.* that

$$\left[m\frac{d}{dt} + m\eta \right] G_R(t) = \delta(t) \tag{10}$$

(b) Show that the solution at time t satisfying the boundary conditions specified at $t = t_o$ are

$$v(t) = mG_R(t, t_o)v(t_o) \tag{11}$$

This is normally how the Green function (propagator) is used in quantum mechanics. The Green function is used slightly differently for second order equations.

Second order: In class we showed that the electric potential can be determined from knowledge of the boundary value and the Green function. A very similar statement can be made about an initial value problem, *i.e.* the solution at future times can be determined from the initial conditions and the Green function.

For definiteness we will take a harmonic oscillator with mass m and resonant frequency ω_o:

$$m\frac{d^2x}{dt^2} + m\omega_o^2 x = 0$$

The retarded Green function $G(t|t_o)$ is the position $x(t)$ of the harmonic oscillator at time t from an impulsive force at time t_o. It is causal, meaning that it vanishes whenever $t < t_o$, *i.e.*

$$\left(m\frac{d^2}{dt^2} + m\omega_o^2 \right) G_R(t|t_o) = \delta(t - t_o) \qquad \text{and } G_R(t, t_o) = 0 \text{ for } t < t_o \tag{12}$$

(a) Given the initial conditions for the oscillator, $x(t_o)$ and $\partial_{t_o} x(t_o)$, at time t_o show that the future value of the oscillator is given by the Wronskian of the Green function and the initial conditions

$$x(t) = m\left[G_R(t, t_o)\partial_{t_o} x_o - x(t_o)\partial_{t_o} G_R(t, t_o) \right] \qquad t > t_o \tag{13}$$

(Hint: closely examine the proof of the Green theorem given in class for the electrostatic case).

(b) Use the Green function for the undamped oscillator given in class to verify that you get the correct result for $x(t)$ in terms of the initial conditions.

(c) Show that for the wave equation, $-\Box G_R(t\boldsymbol{x}|t_o\boldsymbol{x}_o) = \delta(t-t_o)\delta^3(\boldsymbol{x}-\boldsymbol{x}_o)$, the appropriate generalization is

$$u(t, \boldsymbol{x}) = \frac{1}{c^2}\int d^3\boldsymbol{x}_o \left[G(t\boldsymbol{x}|t_o\boldsymbol{x}_o)\partial_{t_o} u(t_o, \boldsymbol{x}_o) - u(t_o, \boldsymbol{x}_o)\partial_{t_o} G(t\boldsymbol{x}|t_o\boldsymbol{x}_o) \right] \tag{14}$$

Remark: The results of this problem show that the general solution to the driven damped harmonic oscillator starting from some initial time moment t_o is

$$\frac{d^2x}{dt^2} + m\eta\frac{dx}{dt} + m\omega_o^2 x(t) = F(t) \tag{15}$$

is

$$x(t) = m\left[G_R(t,t_o)\partial_{t_o}x_o - x(t_o)\partial_{t_o}G_R(t,t_o)\right] + \int_{t_o}^{t} dt'\, G_R(t,t')F(t')\,. \tag{16}$$

At late times (in the presence of any infinitessimal damping) the initial conditions can be ignored.

Similarly for the first order equation:

$$\left[m\frac{d}{dt} + m\eta\right] v(t) = F(t)\,; \tag{17}$$

the general solution is

$$v(t) = mG_R(t,t_o)v(t_o) + \int_{t_o}^{t} dt'\, G_R(t,t')F(t')\,. \tag{18}$$

Problem 6. Green function of the Diffusion equation

Consider the homogeneous diffusion equation:

$$\partial_t n - D\nabla^2 n(t,\boldsymbol{r}) = 0\,. \tag{19}$$

The retarded Green function of the equation satisfies

$$\left[\partial_t - D\nabla^2\right] G(t\boldsymbol{r}|t_o\boldsymbol{r}_o) = \delta(t-t_o)\delta^3(\boldsymbol{r}-\boldsymbol{r}_o)\,. \tag{20}$$

with retarded boundary conditions.

(a) Write Eq. (20) in time and $\boldsymbol{k}$ by introducing the spatial Fourier transform

$$G(t,\boldsymbol{k}) \equiv \int d^3\boldsymbol{r}\, e^{-i\boldsymbol{k}\cdot\boldsymbol{r}} G(t,\boldsymbol{r})\,, \tag{21}$$

and then determine the retarded Green function of the diffusion equation in $\boldsymbol{k}$ and time.

(b) Determine the retarded Green function in ω and $\boldsymbol{k}$, $G_R(\omega,\boldsymbol{k})$, by Fourier transforming Eq. (20) in time and space. Verify that if you perform the Fourier integral over ω that you get the result of part (a).

(c) By taking the spatial Fourier transform verify that

$$G_R(\tau,\boldsymbol{r}) = \theta(\tau)\frac{1}{\sqrt[3]{2\pi\sigma^2(\tau)}}\exp\left(-\frac{(\boldsymbol{r}-\boldsymbol{r}_o)^2}{2\sigma^2(\tau)}\right) \tag{22}$$

where $\sigma^2(t) = 2D\tau$ where $\tau = t - t_o$

Problem 7. Decay of magnetic fields in a sphere

An insulated coil is wound on the surface of a sphere of radius a in such a way to produce a uniform magnetic induction B_o in the z direction inside the sphere and a dipole field outside the sphere. The medium inside and outside the sphere has a uniform conductivity σ and permeability μ.

(a) Find the necessary surface current density $\boldsymbol{K}$, and show that in the vector potential has only an azimuthal component, given by

$$A_\phi = \frac{B_o a^2}{2} \frac{r_<}{r_>^2} \sin\theta \tag{23}$$

where $r_<$ ($r_>$) is the smaller (larger) of r and a.

(b) At $t = 0$ the current in the coil is cut off (the coil's presence may be ignored from now on). Show that, with the neglect of Maxwell's displacement current, the decay of the magnetic field is described by the diffusion equation:

$$\nabla^2 \boldsymbol{A} = \frac{\mu\sigma}{c^2} \frac{\partial \boldsymbol{A}}{\partial t} \,. \tag{24}$$

(c) Using the $(t, \boldsymbol{k})$ green function of the previous problem, show that the vector potential at times $t > 0$ is given by

$$A_\phi = \frac{3B_o a}{\pi} \sin\theta \int_0^\infty e^{-\nu t k^2} j_1(k) j_1(\frac{kr}{a})\, dk \tag{25}$$

where $\nu = c^2/\mu\sigma a^2$ is a characteristic decay rate and $j_1(x) = \sin(x)/x^2 - \cos(x)/x$ is the spherical Bessel function of order one.

(d) Show that the magnetic field at the center of the sphere can be written explicitly in terms of the error function

$$B_z(0,t) = B_o \left[\Phi(\frac{1}{\sqrt{4\nu t}}) - \frac{1}{\sqrt{\pi \upsilon t}} \exp\left(\frac{-1}{4\nu t}\right) \right] \tag{26}$$