Problem 1. A Charged Rotor: 20.25

Problem 2. Radiation from a Phased Array: 20.15

Problem 3. An Uncharged Rotor: 20.20

Problem 4. Hydrogen transitions

The transitional charge and current densities for the radiative transition from the \( m = 0, 2p \) state in hydrogen to the 1s ground state are, with the neglect of spin:

\[
\rho(r, \theta, \phi, t) = e \Psi_{1s}^{\dagger} \Psi_{2p} \\
= \frac{2e}{\sqrt{6a_o^3}} r e^{-3r/2a_o} Y_{00} Y_{10} e^{-i\omega_o t}
\]

\[
J(r, \theta, \phi, t) = \frac{1}{2} e \left[ \Psi_{1s}^{\dagger} \left( \frac{p}{m} \Psi_{2p} \right) - \left( \frac{p}{m} \Psi_{1s}^{\dagger} \right) \Psi_{2p} \right] \\
= -\frac{iv_0}{2} \left( \frac{\hat{r}}{2} + \frac{a_o}{z} \hat{z} \right) \rho(r, \theta, \phi, t)
\]

where \( a_o = 0.529 \text{Å} \) is the Bohr radius, and

\[
h\omega_o = \frac{e^2}{2(4\pi a_o)} \frac{3}{4} \approx 13.6 \text{eV}
\]

is the frequency difference of the levels, and

\[
\beta = \frac{v_o}{c} = \frac{e}{4\pi \hbar c} = \alpha \approx \frac{1}{137}
\]

is the Bohr orbit speed.

(a) Use \( \hbar c = 197 \text{eV} \cdot \text{nm} \) to evaluate the frequency \( \omega_o \) in 1/s.

(b) In the Bohr model an electron in the \( n \)-th orbit circles the proton with angular momentum \( |L| = n \hbar \). Show that the kinetic energy, \( p^2/2m \), is (minus) one half of the potential energy.

Then, establish that if \( |L| = \hbar \) (the \( n = 1 \) Bohr orbit)

\[
\frac{1}{2} mc^2 \alpha^2 = \frac{\hbar^2}{2ma_o^2} = \frac{e^2}{2(4\pi a_o)} = 13.6 \text{eV}
\]

two ways to write KE minus half PE

Remark. This is well worth memorizing and is how I remember the Bohr radius, \( p = \hbar/a_o = mc\alpha \). I recognize that the ground state is reached when the kinetic energy associated with the uncertainty principle \( \sim \hbar^2/(2ma_o^2) \) is balanced by (half) the potential energy \( \sim e^2/(2(4\pi a_o)). \)
(c) Show that the wavelength of the light which is emitted is

\[ k^{-1} = \frac{\lambda}{2\pi} = \frac{8}{3} \frac{a_o}{\alpha} \]  

(8)

and explain why this justifies the multipole expansion.

(d) In the electric dipole approximation calculate the total time-averaged power radiated. Express your answer in units of \(\hbar \omega_o (\alpha^4 c/a_o)\).

(e) Interpreting the classically calculated power as the photon energy \(\hbar \omega_o\) times the transition probability per time (\(\equiv \Gamma\)), determine \(\Gamma/\omega_o\) as a function of \(\alpha\). Evaluate your result for \(\Gamma/\omega_o\) numerically, and evaluate the lifetime \(\equiv 1/\Gamma\) in seconds.

(f) If instead of the semi-classical charge density used above (which gives the correct answer), the electron in the 2p state was described by the \(n = 2\) circular Bohr orbit (i.e. rotating with the orbital velocity and radius of the \(n = 2\) orbit, \(\beta_n = \alpha/n\) and \(r_n = a_o n^2\)) what would the radiated power be? Express your answer in the same units as part (d), and evaluate the ratio of the two powers numerically.

**Problem 5. In class exercise on quadrupole integrals**

In class we showed that the electric field radiated from a quadrupole is

\[ E(t, r) = \frac{-1}{12\pi r c^3} \left[ \Theta \cdot n - n \left( n^T \cdot \Theta \cdot n \right) \right]_{\text{ret}} \]  

(9)

where we have used a matrix notation, and the ret indicates that the quadrupole moment is to be evaluated at \(t - r/c\).

(a) By squaring the electric field and integrating over the angles of \(n\) show that the total power radiated is

\[ P_{E2} = \frac{1}{180\pi c^5} \left[ \Theta_{ab} \Theta^{ab} \right]_{\text{ret}} \]  

(10)

Be explicit about your steps.