Problem 1. A Charged Rotor: 20.25

Problem 2. Radiation from a Phased Array: 20.15

Problem 3. An Uncharged Rotor: 20.20

Problem 4. Hydrodogen transitions

The transitional charge and current densities for the radiative transition from the $m = 0$, $2p$ state in hydrogen to the $1s$ ground state are, with the neglect of spin:

$$\begin{aligned} \rho(r,\theta,\phi,t) =& e\Psi_{1s}^{\dagger}\Psi_{2p} && (1)\\ =& \frac{2e}{\sqrt{6}a_o^4} r e^{-3r/2a_o} Y_{00} Y_{10} e^{-i\omega_o t} && (2)\\ \boldsymbol{J}(r,\theta,\phi,t) =& \tfrac{1}{2}e\left[\Psi_{1s}^{\dagger}\left(\frac{\boldsymbol{p}}{m}\Psi_{2p}\right) - \left(\frac{\boldsymbol{p}}{m}\Psi_{1s}^{\dagger}\right)\Psi_{2p}\right] && (3)\\ =& \frac{-iv_0}{2}\left(\frac{\hat{\boldsymbol{r}}}{2} + \frac{a_o}{z}\hat{\boldsymbol{z}}\right)\rho(r,\theta,\phi,t) && (4) \end{aligned}$$

where $a_o = 0.529\,\mathring{A}$ is the Bohr radius, and

$$\hbar\omega_o = \underbrace{\frac{e^2}{2(4\pi a_o)}}_{\simeq 13.6\,\mathrm{eV}} \frac{3}{4} \tag{5}$$

is the frequency difference of the levels, and

$$\beta = \frac{v_o}{c} = \frac{e^2}{4\pi\hbar c} = \alpha \simeq \frac{1}{137} \tag{6}$$

is the Bohr orbit speed.

(a) Use $\hbar c = 197\,\mathrm{eV}\cdot\mathrm{nm}$ to evaluate the frequency ω_o in 1/s.

(b) In the Bohr model an electron in the n-th orbit circles the proton with angular momentum $|\boldsymbol{L}| = n\,\hbar$. Show that the kinetic energy, $p^2/2m$, is (minus) one half of the potential energy.

Then, establish that if $|\boldsymbol{L}| = \hbar$ (the $n = 1$ Bohr orbit)

$$\underbrace{\frac{1}{2}mc^2\alpha^2 = \frac{\hbar^2}{2ma_0^2}}_{\text{two ways to write KE}} = \underbrace{\frac{e^2}{2(4\pi a_o)}}_{\text{minus half PE}} = 13.6\,\mathrm{eV} \tag{7}$$

Remark. This is well worth memorizing and is how I remember the Bohr radius, $p = \hbar/a_o = mc\,\alpha$. I recognize that the ground state is reached when the kinetic energy associated with the uncertainty principle $\sim \hbar^2/(2ma_o^2)$ is balanced by (half) the potential energy $\sim e^2/2(4\pi a_o)$.

(c) Show that the wavelength of the light which is emitted is

$$k^{-1} = \frac{\lambda}{2\pi} = \frac{8}{3}\frac{a_o}{\alpha} \tag{8}$$

and explain why this justifies the multipole expansion.

(d) In the electric dipole approximation calculate the total time-averaged power radiated. Express your answer in units of $\hbar\omega_o\,(\alpha^4 c/a_o)$.

(e) Interpreting the classically calculated power as the photon energy $\hbar\omega_o$ times the transition probability per time ($\equiv \Gamma$), determine Γ/ω_o as a function of α. Evaluate your result for Γ/ω_o numerically, and evaluate the lifetime $\equiv 1/\Gamma$ in seconds.

(f) If insted of the semi-classical charge density used above (which gives the correct answer), the electron in the $2p$ state was described by the $n = 2$ circular Bohr orbit (*i.e.* rotating with the orbital velocity and radius of the $n = 2$ orbit, $\beta_n = \alpha/n$ and $r_n = a_o n^2$) what would the radiated power be? Express your answer in the same units as part (d), and evaluate the ratio of the two powers numerically.

Problem 5. In class excercise on quadrupole integrals

In class we showed that the electric field radiated from a quadrupole is

$$\boldsymbol{E}(t,\boldsymbol{r}) = \frac{-1}{12\pi r c^3}\left[\dddot{\Theta}\cdot\boldsymbol{n} - \boldsymbol{n}\left(\boldsymbol{n}^T\cdot\dddot{\Theta}\cdot\boldsymbol{n}\right)\right]_{\rm ret} \tag{9}$$

where we have used a matrix notation, and the ret indicates that the quadrupole moment is to be evaluated at $t - r/c$.

(a) By squaring the electric field and integrating over the angles of $\boldsymbol{n}$ show that the total power radiated is

$$P_{E2} = \frac{1}{180\pi c^5}\left[\dddot{\Theta}_{ab}\dddot{\Theta}^{ab}\right]_{\rm ret} \tag{10}$$

Be explicit about your steps.