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Classical Mechanics of Charged Particles

$$\frac{d\vec{p}}{dt} = \vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$$

- Consider the fields specified, and solve for the motion of charged particles.

- This is not really E+M, it is just classical mechanics, we want to solve for the fields.

+special relativity

①

Units and Maxwell Eqs.

a) $\nabla \cdot \vec{E} = \rho/\varepsilon_0$ (Gauss Law)

b) $$\nabla \times \vec{B} = \underbrace{\mu_0 \vec{j}}_{\text{Ampères law}} + \underbrace{\mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}}_{\text{Maxwell correction}}$$

c) $\nabla \cdot \vec{B} = 0$ (No magnetic charge)

d) $\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Lenz' law and Back emf)

Fundamental Problem of Electrodynamics:

- Specify the charges and currents $\rho(x,t)$ and $\vec{j}(x,t)$ and solve for the fields. But respect continuity

$$\partial_t \rho + \vec{\nabla} \cdot \vec{j} = 0$$

②

Qualitative Features:

a) $Q \leftrightarrow E$ charges make E-fields
(Gauss Law)

(Ampère) b) $\vec{J} \leftrightarrow \vec{B}$ moving charges (currents) make B-fields

(Lenz) d) Changing B-fields make induced E-fiel

• Changing B fields are caused by changing currents or accelerating charges

(Maxwell) b) Changing E fields make changing B
Correction
fields, wich in turn makes changing B etc.

This sets off a wave of light where changing changing $\vec{E}$ makes changing $\vec{B}$ and vice versa:

$$c = \frac{1}{\sqrt{\mu_0 \varepsilon_0}}$$

Remarks

1. To get the whole process started you need accelerating charges:

Formula: (worth memorizing!)

$$P = \frac{2}{3}\left(\frac{e^2}{4\pi\varepsilon_0}\right)\frac{1}{c^3}a^2$$

power radiated by an accelerating charge: P

charge: e

acceleration of charges: a

2. We specify currents/charges and solve for fields. (The backreaction of the fields on the currents is not part of classical E+M. But is a part of quantum electrodynamics

Media – we will often study fields in media, where the currents and charges are a function of the fiel[ds thr]ough constituent relations e.g. for a conductor

$$\vec{J} = \sigma\vec{E}$$

current: $\vec{J}$

conductivity: σ

electric field: $\vec{E}$

- Such constituent relations depend on the medium, and are ~~usually~~ valid only when the wavelength is very long compared to micro scales
- For each medium, find a different set of equations to be solved, e.g. for a conductor

$$\nabla \times B = \mu_0 j + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

$$\nabla \times B = \mu_0 \sigma E + \mu_0 \varepsilon_0 \frac{\partial E}{\partial t}$$

Now we solve this equation for the fields, with no reference to the currents.

E&M

- So when asked a question about a medium (conductor, dielectric, topological insulator, ferromagnet) we should first be clear about what the constituent relations are!

→ Then solve the eqns for fields

→ Then you know the currents from constituent relatio.

$$j = \sigma E$$

Units (Use Heavyside Lorentz Units)

$$\nabla \cdot \vec{E} = \rho/\varepsilon_0 \qquad \vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{Q}{r^2}$$

Choose a set of units with $\varepsilon_0 = 1$, then $\mu_0 = \frac{1}{c^2}$

Formally

Define: $\bar{E} = \sqrt{\varepsilon_0}\, E \qquad \bar{Q} = Q/\sqrt{\varepsilon_0} \qquad \bar{\rho} = \rho/\sqrt{\varepsilon_0}$

$\bar{B} = \sqrt{\varepsilon_0}\, B$

Eqs become

$$\nabla \cdot \bar{E} = \bar{\rho}$$

$$\nabla \times \bar{B} = \frac{1}{c^2}\left(j + \frac{\partial \bar{E}}{\partial t}\right)$$

$$\nabla \cdot \bar{B} = 0$$

$$\nabla \times \bar{E} = -\frac{\partial \bar{B}}{\partial t}$$

Then drop bars ... Same as setting $\varepsilon_0 = 1$

- As we will see $\bar{E}$ and $\bar{B}$ are very much the same thing. Define $E_{HL} = \bar{E}$ $B_{HL} = c\bar{B}$ which have same units:

$$E_{HL} \equiv \bar{E} \qquad B_{HL} \equiv c\bar{B}$$

$$= E_{MKS}\sqrt{\varepsilon_0} \qquad B_{HL} = c\, B_{MKS}\sqrt{\varepsilon_0} = \frac{B_{MKS}}{\sqrt{\mu_0}}$$

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And

$$\nabla \cdot E_{HL} = \rho_{HL}$$

$$\nabla \times B_{HL} = \frac{j_{HL}}{c} + \frac{1}{c}\frac{\partial E_{HL}}{\partial t}$$

$$\nabla \cdot B_{HL} = 0$$

$$\nabla \times E_{HL} = -\frac{1}{c}\frac{\partial B_{HL}}{\partial t}$$

We will use this, and will stop writing HL unless needed.

Remarks: (see note)

1. To convrt from MKS to HL set $\varepsilon_0 = 1$ and arrange to multiply all magnetic quantities by c:

$$B_{HL} = c B_{mks}$$

$$m_{HL} = c\, m_{mks} \quad \text{(magnetic moment)}$$

$$A_{HL} = c\, A_{mks} \quad \text{(vector potential)}$$

Ex
$$F = q(E_{mks} + v \times B_{mks})$$

$$= q\left(E_{mks} + \frac{v}{c} \times c B_{mks}\right) = q_{HL}\left(E_{HL} + \frac{v}{c} \times B_{HL}\right)$$

I. HEAVSIDE LORENZ (HL) UNITS

A. MKS to HL Units

- To convert from MKS to HL set $\epsilon_o = 1$ (and thus $\mu_o = 1/c^2$) and recognize that all (*magnetic* MKS quantities)$\times c$ are the corresponding HL quantities. (Some magnetic quantities we will study are the magnetic field, the vector potential, and the magnetic moment, $\mathbf{B}, \mathbf{A}, \boldsymbol{m}$)

  Example: the magnetic potential energy

$$U_B = \frac{1}{2}\frac{B_{MKS}^2}{\mu_o} \Rightarrow \frac{1}{2}(cB_{MKS})^2 = \frac{1}{2}B_{HL}^2 \tag{1.1}$$

  Example: The poynting vector

$$S = \frac{1}{\sqrt{\mu_o}}\mathbf{E}_{MKS} \times \mathbf{B}_{MKS} \Rightarrow \mathbf{E}_{MKS} \times (c\mathbf{B}_{MKS}) = \mathbf{E}_{HL} \times \mathbf{B}_{HL} \tag{1.2}$$

  Example: The force law

$$\boldsymbol{F} = q_{MKS}(\mathbf{E}_{MKS} + \mathbf{v} \times \mathbf{B}_{MKS}) = q_{MKS}(\mathbf{E}_{MKS} + \frac{\mathbf{v}}{c} \times c\mathbf{B}_{MKS}) = q_{HL}(\mathbf{E}_{HL} + \frac{\mathbf{v}}{c} \times \mathbf{B}_{HL}) \tag{1.3}$$

B. HL to MKS

- The relation between charges and and currents in the HL and MKS units are

$$Q_{HL} = \frac{Q_{MKS}}{\sqrt{\epsilon}} \qquad \rightarrow \qquad \frac{1}{\sqrt{\epsilon_o}}\,(1\,\mu\text{C}) = 0.336\,\sqrt{N \cdot m^2} \tag{1.4}$$

$$\frac{I_{HL}}{c} = \frac{I_{MKS}}{\sqrt{\epsilon}c} = \sqrt{\mu_o}I \qquad \rightarrow \qquad \sqrt{\mu_o}(1\,\text{amp}) = 0.00112\,\sqrt{N \cdot m^2} \tag{1.5}$$

- The relation between Field strengths and is

$$E_{HL} = \sqrt{\epsilon_o}\,E_{MKS} \qquad \rightarrow \qquad \sqrt{\epsilon_o}\,(1\,\text{kV/cm}) = 0.2975\,\sqrt{N/m^2} \tag{1.6}$$

$$B_{HL} = \sqrt{\epsilon_o}\,(cB_{MKS}) = \frac{1}{\sqrt{\mu_o}}B_{MKS} \qquad \rightarrow \qquad \frac{1}{\sqrt{\mu_o}}(1\,\text{Tesla}) = 892.062\,\sqrt{N/m^2} \tag{1.7}$$

Dimensional Analysis of Maxwell Eq.s

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\frac{\partial E}{\partial t}$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

The speed of Light is fast compared to macroscopic scales. Terms with $1/c$ are smaller. Experiments have a characteristic length scale, L, time scale, T, and charge, Q. Formally define:

$$\bar{t} \equiv \frac{t}{T} \qquad \bar{x} \equiv \frac{x}{L} \qquad \nabla \equiv \frac{1}{L}\bar{\nabla}$$

$$\equiv \frac{1}{L}\left(\hat{x}\frac{\partial}{\partial \bar{x}} + \hat{y}\frac{\partial}{\partial \bar{y}} + \hat{z}\frac{\partial}{\partial \bar{z}}\right)$$

Similarly

$$\rho \equiv (Q/L^3)\,\bar{\rho}$$

$$j \equiv \left(\frac{Q}{L^2 T}\right)\bar{j}$$

and most importantly

$\bar{c} \equiv \dfrac{c}{(L/T)} \gg 1$, since c is fast; $\bar{c} \sim 10^{8}$

$\dfrac{1}{\bar{c}} \sim 10^{-8}$

Then

$$\bar{\nabla} \cdot \bar{E} = \bar{\rho}$$

now we will stop writing the bar

$$\bar{\nabla} \times \bar{B} = \frac{1}{\bar{c}} \bar{J} + \frac{1}{\bar{c}} \frac{\partial \bar{E}}{\partial \bar{t}}$$

$$\bar{\nabla} \cdot \bar{B} = 0$$

$$\bar{\nabla} \times \bar{E} = -\frac{1}{\bar{c}} \frac{\partial \bar{B}}{\partial \bar{t}}$$

And set up a series in $1/c$:

$$E = E^{(0)} + E^{(1)} + E^{(2)} + \dots$$

$$B = B^{(0)} + B^{(1)} + B^{(2)} + \dots$$

i.e. $E^{(1)}$ is of order $\dfrac{1}{\bar{c}} E^{(0)} \sim 10^{-8} E^{(0)}$;

$E^{(2)}$ " " " $\dfrac{1}{\bar{c}^2} E^{(0)} \sim 10^{-16} E^{(0)}$

At zeroth order

$$\nabla \cdot E^{(0)} = \rho$$

$$\nabla \times B^{(0)} = 0$$

$$\nabla \cdot B^{(0)} = 0$$

$$\nabla \times E^{(0)} = 0$$

$\Rightarrow$ Find $B^{(0)} = 0$ and

$$\nabla \cdot E^{(0)} = \rho$$

$$\nabla \times E^{(0)} = 0$$

This is the equations of electrostatics.

At first order:

$$\nabla \cdot E^{(1)} = 0$$

$$\nabla \times B^{(1)} = \frac{j}{c} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t}$$

$$\nabla \cdot B^{(1)} = 0$$

$$\nabla \times E^{(1)} = -\frac{1}{c}\cancel{\frac{\partial B^{(0)}}{\partial t}}^{\,0}$$

Find $E^{(1)} = 0$ and

computed with electro statics

$$\nabla \times B^{(1)} = \frac{j}{c} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t}$$

$$\nabla \cdot B^{(1)} = 0$$

this is magneto statics with Maxwel displacement current

Can continue in this way, and work out systematic corrections to electric and magneto statics ....

We will come to this Later ....