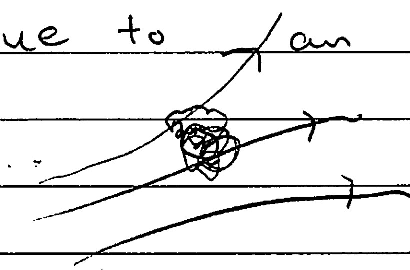
Forces on charge distributions:

- Before we calculated the field far from a localized charge distribution in terms of multipoles

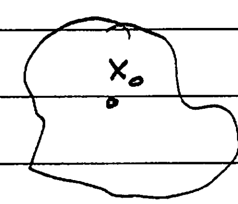
- Now we will calculate force on a small object due to an external field



Solution $F_i = -\partial_i U(x_0)$ where

$$\boxed{U(x_0) = Q_V \, \varphi(x_0) - \vec{p}\cdot E(x_0) - \frac{1}{3}\Theta^{ij}\partial_i E_j(x_0) + \dots}$$

Where



$$Q = \int_V \rho(\vec{r})$$

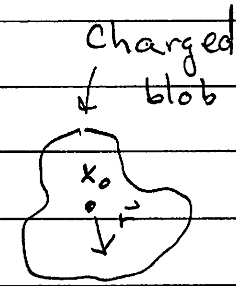
$$\vec{p} = \int \rho(\vec{r})\, \delta\vec{r} \qquad \delta\vec{r} \equiv \vec{r} - \vec{x}_0$$

$$\Theta^{ij} = \int \rho(r)\left[\frac{3}{2}\delta\vec{r}^{\,i}\delta r^j - \frac{1}{2}\delta\vec{r}^{\,2}\delta^{ij}\right]$$

Are the usual multipoles

Prf

Pick a point in the body. Expand the electric field arond that point



potential w/out the charged blob

$$\star \quad U = \int \rho(\vec{r})\, \varphi_{ext}(\vec{r}) \qquad \nabla \cdot E = 0$$

$$\varphi = \varphi(x_0) + \delta r^i \partial_i \varphi + \frac{1}{2} \partial_i \partial_j \varphi\, \delta r^i \delta r^j + \dots$$

Note since $\nabla \cdot E = 0$ or

$-\partial_i \partial^i \varphi = 0$ and $\partial_i \partial_j \varphi$ is traceless

δ^{ij} Contracts these

Thus

$$\frac{1}{2} \partial_i \partial_j \varphi\, \delta r^i \delta r^j = \frac{1}{2} (\partial_i \partial_j \varphi) \left(\delta r^i \delta r^j - \frac{1}{3} \delta r^2 \delta^{ij} \right)$$

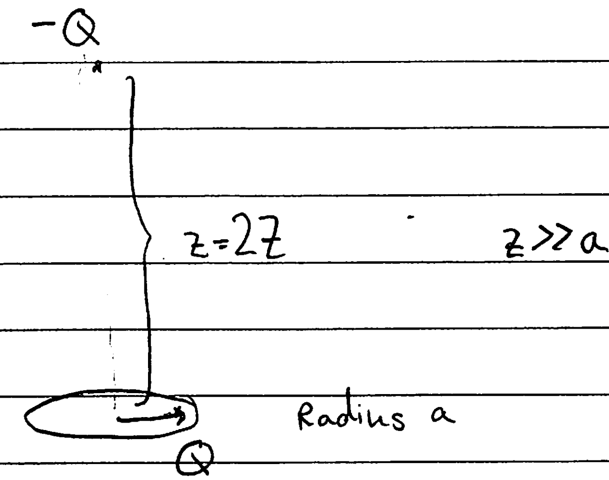
$$= \frac{1}{3} (\partial_i \partial_j \varphi) \left(\frac{3}{2} \delta r^i \delta r^j - \frac{1}{2} \delta r^2 \delta^{ij} \right)$$

Plugging into $\star$

$$U(x_0) = \int_V \rho(\vec{r}) \left[\varphi(x_0) - \delta r^i \cdot \vec{E}(x_0) - \frac{1}{3} (\partial_i E_j) \left(\frac{3}{2} \delta r^i \delta r^j - \frac{1}{2} \delta r^2 \delta^{ij} \right) + \dots \right]$$

Ring Problem (pg. 1)

Problem: Use this



The homework shows that the potential due to the ring is:

$$\star \quad \varphi \simeq \frac{Q}{4\pi r} - \frac{1}{2}\frac{Qa^2}{4\pi r^3} P_2(\cos\theta) + \dots$$

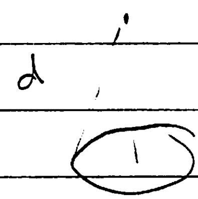
Determine the cartesian components of Θ_{ij} P_{ij} and determine the Force on the ring due to the point charge two ways

- Elementary way
- Show that in a general field $\varphi(z)$

$$U_{int} = Q\,\varphi(z) - \frac{1}{2}\Theta^{zz}\,\partial_z E_z(z)$$

Ring Problem (pg. 2)

① Elementary way



$$u = \frac{-Q^2}{4\pi d} = \frac{-Q^2}{4\pi (z^2 + a^2)^{1/2}}$$

$$\simeq \frac{-Q^2}{4\pi z} + \frac{1}{2}\frac{Q^2 a^2}{4\pi z^3} + \dots$$

② Using the cartesian expansion for the potential far from ring.

$$☆☆ \quad \varphi(\vec{r}) = \frac{Q}{4\pi r} + \frac{p^i \hat{r}_i}{r^2} + \frac{\Theta^{ij}}{r^3}\hat{r}_i \hat{r}_j$$

Comparison ω ☆ shows $p^i = 0$ (see previous page)

Since ☆ has no $1/r^2$ terms

Now the potentials are equal:

$$\frac{\Theta^{ij}\hat{r}_i \hat{r}_j}{r^3} = \frac{-1}{2}\frac{Q a^2}{4\pi r^2}\underbrace{\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}_{P_2(\cos\theta)}$$

Choose

Θ_{ij} so that these are equal.

Ring Problem (pg. 3)

Since

$$\hat{r} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$$

The most general form of Θ_{ij} is

$$\begin{pmatrix} -\frac{\Theta_{zz}}{2}+\Delta & \Theta_{xy} & \Theta_{xz} \\ \Theta_{xy} & -\frac{\Theta_{zz}}{2}-\Delta & \Theta_{yz} \\ \Theta_{xz} & \Theta_{yz} & \Theta_{zz} \end{pmatrix}$$

We want no ϕ dependence. Thus the only possible form is:

$$\Theta_{ij} = \begin{pmatrix} -\Theta_{zz}/2 & & \\ & -\Theta_{zz}/2 & \\ & & \Theta_{zz} \end{pmatrix}$$

$\Theta_{xy}\hat{r}^x\hat{r}^y \sim \Theta_{xy}\sin^2\theta \cdot (\sin\phi\cos\phi)$

$$\Theta_{ij}\hat{r}^i\hat{r}^j = \left(-\frac{\sin^2\theta}{2} + \cos^2\theta\right)\Theta_{zz}$$

$$= \left(-\frac{1}{2} + \frac{3}{2}\cos^2\theta\right)\Theta_{zz}$$

$$\Theta_{ij}\hat{r}^i\hat{r}^j = P_2(\cos\theta)\,\Theta_{zz}$$

So comparison with Eq ☆ and ☆☆ shows (two pgs. back)

$$\Theta_{zz} = -\frac{1}{2}Qa^2 \qquad \Theta_{xx} = \Theta_{yy} = -\frac{\Theta_{zz}}{2} = \frac{Qa^2}{4}$$

Ring Problem (pg. 4)

The potential energy of a ring in a field

$$U = Q\varphi - \frac{1}{3}\Theta^{ij}\partial_i E_j$$

$$U = Q\varphi - \frac{1}{3}\left(\Theta^{zz}\partial_z E_z - \frac{\Theta^{zz}}{2}\underbrace{(\partial_x E_x + \partial_y E_y)}_{-\partial_z E_z}\right)$$

$$= Q\varphi - \frac{1}{\cancel{3}}\cdot\frac{\cancel{3}}{2}\Theta^{zz}\partial_z E_z$$

$$\boxed{U = Q\varphi - \frac{1}{2}\Theta^{zz}\partial_z E_z}$$

$\downarrow$

Now you can use this.

$$E_z = \frac{-Q}{4\pi z^2} \quad \text{field due to monopole}$$

$$\partial_z E_z = \frac{+2Q}{4\pi z^3}$$

$$U \simeq \frac{-Q^2}{4\pi z} - \frac{1}{2}\left(-\frac{1}{2}Qa^2\right)\left(\frac{2Q}{4\pi z^3}\right) + \cdots$$

$$\boxed{U \simeq -\frac{Q^2}{4\pi z} + \frac{1}{2}\frac{Q^2 a^2}{4\pi z^3}} \longrightarrow \text{agrees (w) elementary way}$$

Problem Taxonomy

- Got changes (?) charged shells etc
  - → Green fcn (
  - → Separation of variables. But you will need to analyze jump conditions (spherical shell chrg. rings)

- External Fields. Try subtracting it off. Not necessary but useful. (Defects in capacitors)

- No Charges. Just Boundary vals
  - → Do separation. Treat two coords parallel to surface different from perp
  - → Or use Grn fcn technique,

$$\varphi(r) = \cancel{\int_V G(r,r_0)\,\rho(r_0)} - \int_{S_0} ds_0\, \vec{n} \cdot \nabla_{r_0} G(r,r_0)\, \varphi_0(r_0)$$

no charges

Need Grn fcn:

① Images. All problems variants of sphere/cylinder/plane.

② Expand two dimension in efcns and solve for remaing dimension by integration

③ Expand in 3Dimensions

Grn fcn

An alternate $\vee$ method for sphere phoblem

$$\varphi(\vec{r}) = \int \rho(r_0)\, G(\vec{r}, r_0)$$

$$G(\vec{r}, \vec{r}_0) = \frac{1}{4\pi |\vec{r} - \vec{r}_0|}$$

$$\rho(r) = \sigma(\theta, \phi)\, \delta(r - R)$$

$$\varphi(\vec{r}) = \int G(r, \theta\varphi ; R\, \theta_0 \phi_0)\, \sigma(\theta_0, \phi_0) R^2\, d(\cos\theta_0)\, d\phi_0$$

Using

$$\frac{1}{4\pi |\vec{r} - \vec{r}_0|} = \sum_{\ell m} \frac{r_<^{\ell}}{r_>^{\ell+1}}\, Y_{\ell m}(\theta, \phi)\, Y^*_{\ell m}(\theta_0, \phi_0)\, \frac{1}{2\ell+1}$$

and

$$\sigma(\theta_0, \phi_0) = \sigma_0 (\cos\theta_0 + \frac{1}{2} \sin\theta_0 \cos\phi_0)$$

$$\equiv \sum_m \sigma_{1m}\, Y_{1m}(\theta_0, \phi_0)$$

$$\cos\theta = \sqrt{\frac{4\pi}{3}}\, Y_{10} \qquad \frac{1}{2}\sin\theta\cos\phi = \sqrt{\frac{8\pi}{3}} \left(\frac{-Y_{11} + Y_{1-1}}{4} \right)$$

So for $r > R$ $\quad r_> = r \quad r_< = R$

$$\varphi(r) = \int R^2 \, d\Omega_0 \sum_{\ell m m'} \overset{1/2\ell+1}{\checkmark} \frac{R^\ell}{r^{\ell+1}} Y_{\ell m}(\theta,\phi) \overbrace{Y^*_{\ell m}(\theta_0,\phi_0) \, Y_{1m'}(\theta_0,\phi_0)}\ \sigma_{1m'}$$

Integration only $\ell = 1$ $m = m'$ survivus: Since

$$\underbrace{\int d\Omega_0 \, Y^*_{\ell m}(\theta_0,\phi_0) \, Y_{1m'}(\theta_0,\phi_0)} = \delta_{\ell 1} \delta_{m m'}$$

So

$$\varphi(r) = \frac{1}{3} \frac{\left(\sum_{m'} \sigma_{1m'} Y_{1m'}(\theta,\phi) \right) R^3}{r^2}$$

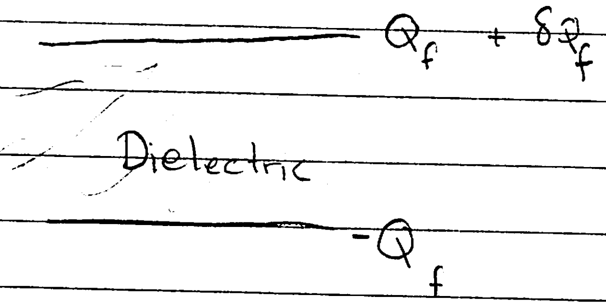
$$\varphi(\vec{r}) = \frac{1}{3} \frac{\sigma(\theta,\phi) R^3}{r^2}$$

Similarly, for $r < R$ $\quad r_> = R \quad r_< = r$

$$\varphi(\vec{r}) = \frac{1}{3} \sigma(\theta,\phi) \, r$$

↳ ~~Work it out yourself~~

Energy in Dielectrics



The energy required to add dQ_f

$$\delta W = \int_V \delta\rho_f \cdot \varphi$$

$\leftarrow$ External work required to charge capacitor including to polarize dielectric (See Griffiths)

$$= \int_V (\nabla \cdot \delta\vec{D})\, \varphi$$

$$= -\int_V \delta\vec{D} \cdot \vec{\nabla}\varphi$$

$$\delta W = \int_V \delta D^{(E)} \cdot \vec{E} \implies W = \int_V \int_0^D \vec{E}(D) \cdot d\vec{D}$$

For linear substance

$$\delta\vec{D} = \varepsilon(r)\, \vec{E}$$

$$\delta W = \int_V \varepsilon(r)\, \delta E \cdot E \implies W = \int_V \frac{1}{2} \varepsilon(r)\, \vec{E}^2(r)$$

Energy in Dielectrics Pg. 2

So

$$\boxed{W = \frac{1}{2}\int_V E \cdot \vec{D}}$$

For uniform substance (in thermo)

$$\boxed{dW = V\, \vec{E} \cdot d\vec{D}}$$

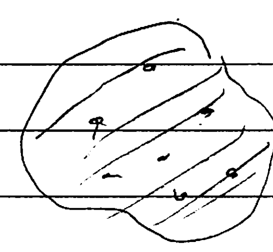
Forces and Stress Tensor

- Focus on material with $\varepsilon = \text{const}$
  (Can treat piecewise const with this)

$\vec{f} = \rho_f \vec{E}$ (force per volume)

Will show for linear media

$$F^j = -\int dS \; n_i \, T_E^{ij}$$



stress tensor (units) = force per area

$$T_E^{ij} = -E^i D^j + \frac{1}{2} E \cdot D \, \delta^{ij}$$

$$\boxed{T_E^{ij} = \left(-\varepsilon E^i E^j + \frac{1}{2}\varepsilon E^2\right)\delta^{ij}}$$

(Also works for ε not constant see book)

Same as vac by $\varepsilon \neq 1$

Force & Stress Pg. 2

Prf

$$f^j = \rho_f E^j$$

$\nabla \times E = 0 \qquad \partial_i E_j = \partial_j E_i$

$$= (\partial_i D^i) E^j$$

$$= \partial_i (D^i E^j) - D^i \partial_i E^j$$

$$= \partial_i (D^i E^j) - D^i \partial_j E^i$$

linear media
ε const

$$= \partial_i (D^i E^j) - \frac{1}{2} \partial_i (D \cdot E \, \delta^{ij})$$

$\partial_i D = \varepsilon \partial_i E$

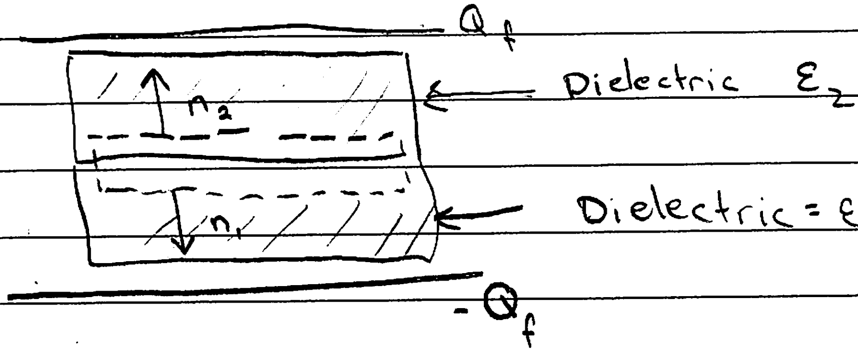
$$f^j = \partial_i \underbrace{\left(D^i E^j - \frac{1}{2} D \cdot E \, \delta^{ij}\right)}_{= -T_E^{ij}}$$

So

$$F^j = \int_V f^j = \int_V -\partial_i T^{ij}$$

$$F^j = -\int dS \, n_i T_E^{ij}$$

Problem



- Calculate the force per area on the interface. Method I; energy. Method 2 stress tensor

$$F^z = -\int_{\text{dased line}} dS \; n_i \bar{T}^{ij}$$

$$= -A\,T_2^{zz} + A\,T_1^{zz}$$

Now

$$-T_2^{zz} = E^z D^z - \frac{1}{2} E \cdot D\, \delta^{zz} = +\frac{1}{2} E^z D^z = \frac{1}{2}\frac{(D_2^z)^2}{\varepsilon_2}$$

and similarly, $T_1^{zz} = \frac{1}{2}\frac{(D_1^z)^2}{\varepsilon_1}$. So

$$F^z = \frac{1}{2}\frac{(D_2^z)^2}{\varepsilon_2} - \frac{1}{2}\frac{(D_1^z)^2}{\varepsilon_1}$$

Using continuity of D at interface, $D_2 - D_1 = \cancel{\sigma_f}^{\,0}$. An at the metal plates $\vec{n}\cdot\vec{D} = \sigma_f$. Yielding

$$\boxed{F^z = \frac{1}{2}\sigma_f^2\left(\frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_1}\right)}$$