

Magneto Statics

$$\nabla \cdot \mathbf{E} = \rho$$

$$\nabla \times \mathbf{B} = \mathbf{j}/c + \frac{1}{c} \partial_t \mathbf{E}$$

$$\nabla \cdot \mathbf{B} = 0$$

$$\nabla \times \mathbf{E} = -\frac{1}{c} \partial_t \mathbf{B}$$

} Full Maxwell eqs

After the $1/c$ expansion

$$\nabla \cdot \mathbf{E}^{(0)} = \rho$$

$$\nabla \times \mathbf{B}^{(0)} = 0$$

$$\nabla \cdot \mathbf{B}^{(0)} = 0$$

$$\nabla \times \mathbf{E}^{(0)} = 0$$

} \Rightarrow solve e-statics for $\mathbf{E}^{(0)}$

$$\mathbf{B}^{(0)} = 0$$

Now at next order

$$\nabla \cdot \mathbf{E}^{(1)} = 0$$

$$\nabla \times \mathbf{B}^{(1)} = \mathbf{j}/c + \frac{1}{c} \partial_t \mathbf{E}^{(0)}$$

$$\nabla \cdot \mathbf{B}^{(1)} = 0$$

$$\nabla \times \mathbf{E}^{(1)} = 0$$

} $\mathbf{E}^{(1)}$ is zero

So find

calculated with electro statics

$$\begin{aligned}\nabla \times \vec{B}^{(1)} &= \vec{j} + \frac{1}{c} \frac{\partial \vec{E}^{(0)}}{\partial t} \\ \nabla \cdot \vec{B}^{(1)} &= 0\end{aligned}$$

Maxwell displacement current

The RHS consists of two known terms

$$\vec{j}_{\text{tot}} = \vec{j} + \frac{\partial \vec{E}^{(0)}}{\partial t}$$

Current displacement current

Very often $\frac{\partial \vec{E}^{(0)}}{\partial t}$ will be zero if there are only currents and no accumulation of charge. Then:

$$\vec{j}_{\text{tot}} = \vec{j}$$

ignoring $\frac{\partial \vec{E}^{(0)}}{\partial t}$

I will continue to write \vec{j} . But, the derivations would go through, if I used \vec{j}_{tot} instead of \vec{j} .

Then magnetostatics is:

This is equivalent to $\nabla \times \vec{B} = \vec{j}/c$

Ampères Law \Rightarrow

and $\nabla \cdot \vec{B} = 0$

and Biot-Savart Law

Law

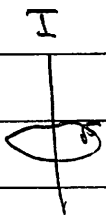
$$\vec{F} = q \frac{\vec{v}}{c} \times \vec{B} \quad \text{and} \quad d\vec{F} = \frac{I}{c} d\vec{l} \times \vec{B}$$

Ampère's Law

$$\oint \vec{B} \cdot d\vec{l} = \int \vec{\nabla} \times \vec{B} \cdot d\vec{a}$$
$$= \int d\vec{a} \cdot \left(\frac{\vec{j}}{c} \right)$$

$$\boxed{\oint \vec{B} \cdot d\vec{l} = \frac{I}{c}}$$

Ex

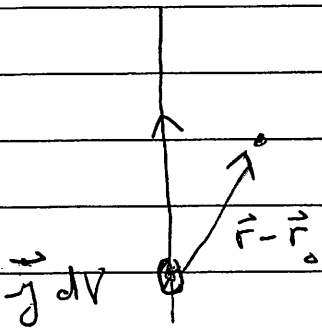


$$2\pi\rho B = I/c \rightarrow B = \frac{I/c}{2\pi\rho}$$

Biot Savart

$$\vec{B} = \int d^3\vec{r} \frac{\vec{j}}{c} \times \left[\frac{(\vec{r} - \vec{r}_0)}{4\pi|\vec{r} - \vec{r}_0|^3} \right]$$

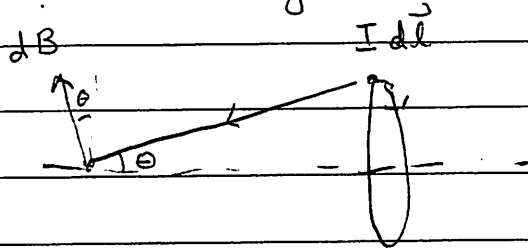
For a wire



$$\vec{B} = \int \frac{I d\vec{l}}{c} \times \frac{(\vec{r} - \vec{r}_0)}{4\pi|\vec{r} - \vec{r}_0|^2}$$

• We will derive this in the next section

Ex: A ring of current Calculate \vec{B} along x-axis



$$dB_x = dB \sin\theta = \frac{I/c \, dl \, \sin\theta}{4\pi (x^2 + a^2)} = \frac{I/c \, dl \, a}{4\pi (x^2 + a^2)^{3/2}}$$

$$B_x = \frac{(I/c \pi a^2)}{4\pi (x^2 + a^2)^{3/2}} \cdot 2$$

$$B_x = \frac{m}{4\pi (x^2 + a^2)^{3/2}} \cdot 2$$

$$m = (I \pi a^2)/c$$

Magnetic moment

$$B_z \xrightarrow{x \rightarrow \infty} \frac{m}{4\pi x^3} \cdot 2$$

we will describe the two later when we describe the multipole expansion

Vector Potential and a proof of Biot-Savart

$$\nabla \times \vec{B} = \vec{j}/c$$

$$\nabla \cdot \vec{B} = 0$$

- Helmholtz theorem: whenever $\nabla \cdot \vec{B} = 0$ \vec{B} can be written as a curl of a vector field

$$\vec{B} = \nabla \times \vec{A}$$

However, the choice is not unique:

$$\vec{A} \rightarrow \vec{A} + \nabla \Lambda \quad \leftarrow \text{Gauge Transform}$$

leads to the same B-field since $\nabla \times (\nabla \Lambda) = 0$,
 $\epsilon^{ijk} \partial_j \partial_k \Lambda = 0$.

- Plugging $\vec{B} = \nabla \times \vec{A}$ into $\nabla \times \vec{B} = \vec{j}/c$

$$\nabla \times (\nabla \times \vec{A}) = \vec{j}/c$$

or using vector identities:

$$\text{Aside } -\nabla^2 \vec{A} + \nabla (\nabla \cdot \vec{A}) = \vec{j}/c$$

$$(\nabla \times (\nabla \times \vec{A}))^i = \epsilon^{ijk} \partial_j (\epsilon_{klm} \partial^l A^m) = \epsilon^{ijk} \epsilon_{klm} \partial_j \partial^l A^m$$

$$= (\delta^i_l \delta^j_m - \delta^i_m \delta^j_l) \partial_j \partial^l A^m$$

$$= \partial^i (\partial_j A^j) - \partial_l \partial^l A^i = [\nabla (\nabla \cdot \vec{A}) - \nabla^2 \vec{A}]^i$$

Vector Potential & Bio-Sawat

So if we adopt the Coulomb Gauge

$$\vec{\nabla} \cdot \vec{A} = 0$$

Which fixes the choice of Λ , then

$$-\nabla^2 \vec{A} = \vec{j}/c$$

Thus the Cartesian components of $\vec{A} = (A^x, A^y, A^z)$ obey the Laplace eqn; e.g.

$$-\nabla^2 A^x = j^x/c$$

So we write down the solution straight away

$$\vec{A}(\vec{r}) = \int d^3x \frac{\vec{j}/c(\vec{x})}{4\pi |\vec{r} - \vec{x}|}$$



kind of coulomb law for magnetic fields

To compute \vec{B} we take the curl:

$$\vec{B} = \nabla_r \times A(\vec{r}) = \int d^3x \vec{\nabla}_r \times \left(\frac{\vec{j}(\vec{x})/c}{4\pi |\vec{r} - \vec{x}|} \right)$$

Vector Potential and Bio-Sawat pg 3

For a constant vector $\vec{j}(x)$ (indep of \vec{r})

$$\vec{\nabla}_x (\vec{j} \phi(r)) = \vec{j} \times (-\vec{\nabla}_r \phi)$$

Yield

$$\vec{B} = \int d^3\vec{x} \vec{j}(x) \times \left(-\nabla \frac{1}{4\pi|\vec{r}-\vec{x}|} \right)$$

gradient of $\frac{1}{r}$

potential

$$= \frac{\hat{r}}{r^2}$$

$$\vec{B} = \int_V d^3x \vec{j}(x) \times \frac{(\vec{r}-\vec{x})}{4\pi|\vec{r}-\vec{x}|^3}$$

$$= \frac{\hat{r}}{r^3}$$

Biot - Savat Law