Last Time

Started to discuss magneto-statics:

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\frac{\partial E}{\partial t} \qquad \text{and} \qquad \partial_t \rho + \nabla \cdot \vec{j} = 0$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\frac{\partial B}{\partial t}$$

After $1/c$ expansion:

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\frac{\partial E^{(0)}}{\partial t} \equiv \frac{j_{TOT}}{c} \quad \text{and} \quad \left(\begin{array}{c} \nabla \cdot E^{(0)} = \rho \\ \nabla \times E^{(0)} = 0 \end{array} \right)$$

$$\nabla \cdot B = 0$$

So

$$\boxed{j_{TOT} = \vec{j} + \frac{\partial \vec{E}^{(0)}}{\partial t}}$$ (Electric field computed with estatics)

$$\nabla \cdot \vec{j}_{TOT} = \nabla \cdot \vec{j} + \frac{\partial}{\partial t} \overbrace{\nabla \cdot E^{(0)}}^{=\rho} = 0$$

$$\boxed{\nabla \cdot \vec{j}_{TOT} = 0}$$

Last time (pg. 2)

So often, $\rho = 0$, and $\partial_t E^{(0)} = 0$, and $\vec{j}_{TOT} = \vec{j}$. Then we find the canonical form

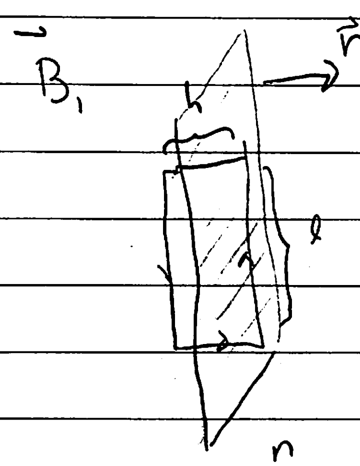
$$\nabla \times \vec{B} = \frac{\vec{j}}{c} \qquad \nabla \cdot \vec{j} = 0$$

$$\nabla \cdot B = 0$$

In order to solve we need boundary conditions ...

Boundary Conditions

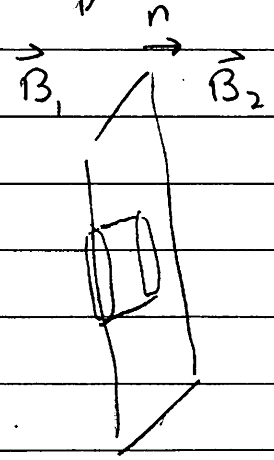
Before we can actually solve the eqns we need B.C



current per area

$$\oint \vec{B}\cdot d\vec{l} = \frac{\vec{K}}{c}\cdot hl$$

$$\boxed{\vec{n}\times(\vec{B}_2-\vec{B}_1) = \frac{\vec{K}}{c}} \Rightarrow B_2^{\perp} - B_1^{\perp} = K$$



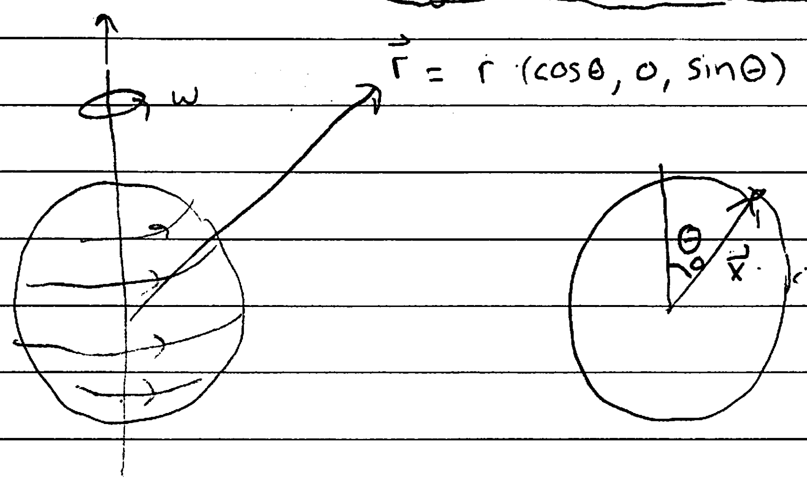
$$\int_{\text{pill box}} (\nabla\cdot B) = 0$$

$$\Rightarrow n\cdot(\vec{B}_2 - \vec{B}_1) = 0$$

$$\boxed{B_2^{\perp} - B_1^{\perp} = 0}$$

- The parallel components jump while the perpendicular components change smoothly

An Example: A rotating charged sphere



radius $\equiv a$

$\vec{x}$: defined with θ_0, ϕ_0

$$\vec{v} = \vec{\omega} \times \vec{x} = \omega a \sin\theta_0 \; \hat{\phi}_0$$

Now:

$$\vec{j} = \rho \vec{v} = \sigma\, \delta(r-a)\, \omega a \sin\theta_0$$

Now solve using the vector potential

$$\nabla \times B = j$$

$$\nabla \cdot B = 0 \Longrightarrow \boxed{\vec{B} = \nabla \times \vec{A}}$$

So

$$\nabla \times (\nabla \times A) = \vec{j}$$

identity $(\nabla^2 \vec{A})^i = \partial_\ell \partial^\ell A^i$

$$-\nabla^2 A + \vec{\nabla}(\nabla \cdot \vec{A}) = \vec{j}$$

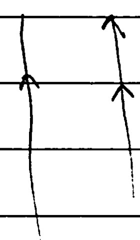
Use coulomb gauge $\boxed{\nabla \cdot A = 0}$

$$\boxed{-\nabla^2 \vec{A} = \vec{j}/c}$$

Methods of Solution:

- Direct use of Eqn. In general coordinate systems $-\nabla^2 \vec{A}$ is a mess, giving rise to three coupled eqns. Only useful for 2D problems with cylindrical symmetry, and no z-dependence

J^z



$$-\nabla^2 A^z = J^z(\rho,\phi)$$

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial \rho}\rho\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \right] A^z = J^z(\rho,\phi)$$

use previous methods to solve for A^z

- Direct integration

$$\vec{A}(\vec{r}) = \int d^3\vec{x}\, \frac{\vec{J}(\vec{x})/c}{4\pi\,|\vec{r}-\vec{x}|}$$

This is clearly the choice here:

$$\vec{A}(\vec{r}) = \frac{1}{c}\int r^2 d\Omega_0\, \frac{\sigma\, \delta(r-a)\, wa\sin\theta_0\, \hat{\phi}_0}{4\pi\,|\vec{r}-\vec{x}|}$$

$$\bigstar \quad \vec{A}(\vec{r}) = \frac{a^2\sigma}{c}\int d\Omega_0\, \frac{1}{4\pi\,|\vec{r}-\vec{x}|}\, \overbrace{wa\sin\theta_0\, \hat{\phi}_0}^{\propto\, Y_{11}(\theta_0,\phi_0) \text{ and } Y_{1-1}(\theta_0,\phi_0)}$$

Rotating Sphere pg. 3

Now look outside $x < r$

$$\frac{1}{4\pi|\vec{r}-\vec{x}|} = \sum_{lm} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}(\theta, \theta) \, Y^*_{lm}(\theta_0, \phi_0)$$

$$= \sum_{lm} \frac{1}{2l+1} \frac{x^l}{r^{l+1}} Y_{lm}(\theta, \phi) \, Y^*_{lm}(\theta_0, \phi_0)$$

Substituting into ☆ we find integrals like

$$\int d\Omega_0 \, Y^*_{lm}(\theta_0, \phi_0) \underbrace{\sin\theta_0}_{Y_{11} \text{ and } Y_{1-1}} = 0 \quad \text{unless } l=1$$

Thus only $l=1$ survives integration.

Comparison with the cartesian form:

$$\frac{1}{4\pi|\vec{r}-\vec{x}|} = \frac{1}{4\pi\sqrt{r^2+x^2-2\vec{r}\cdot\vec{x}}}$$

only this survives

$$= \underbrace{\frac{1}{4\pi r}}_{l=0} + \underbrace{\frac{1}{4\pi r^3}\vec{r}\cdot\vec{x}}_{l=1} + \underbrace{O(x^2)}_{l=2} + \underbrace{O(x^3)}_{l=3} + \dots$$

We see that inside the integral can replace

$$\frac{1}{4\pi|\vec{r}-\vec{x}|} \longrightarrow \frac{1}{4\pi} \frac{\vec{r}\cdot\vec{x}}{r^3}$$

Rotating Sphere pg. 4

So

$$\vec{A} = \frac{a^2\sigma}{c}\int d\Omega_0 \, \frac{1}{4\pi}\,\frac{\vec{r}\cdot\vec{x}}{r^3}\,\overbrace{(\vec{\omega}\times\vec{x})}^{\omega a \sin\theta_0 \hat{\phi}_0}$$

$$\hat{\phi}_0 = \cos\phi_0 \,\hat{x} + \sin\phi_0 \,\hat{y}$$

So,

$$\left[\vec{r}\cdot\vec{x}\,(\vec{\omega}\times\vec{x})\right]^i = \varepsilon^{ijk}\,\omega_j\, x_k\, x_\ell\, r^\ell$$

So

$$A^i = \frac{a^2\sigma\,\varepsilon^{ijk}\omega_j r^\ell}{c\,4\pi r^3}\underbrace{\int d\Omega_0\, x_k x_\ell}_{\frac{4\pi a^2}{3}\delta_{k\ell}}$$

Now

$$\int d\Omega_0\, x_k x_\ell = 0 \quad k\neq\ell \quad \text{e.g.} \int_{\text{sphere}} yz = 0$$

i.e. y^2 i.e. z^2

$$\left.\begin{aligned} &= \int d\Omega_0\, x_1^2 = \int d\Omega_0\, x_2^2 = \int d\Omega\, x_3^2 \\ &= \frac{1}{3}\int d\Omega_0\, \vec{x}\cdot\vec{x} \quad (x_1^2+x_2^2+x_3^2) \\ &= \frac{1}{3}\,4\pi a^2\,\delta_{k\ell} \end{aligned}\right\} \text{for } k=\ell$$

Rotating Sphere pg. 4½

Leading to result that:

$$A^i = \frac{a^2 \sigma}{c\, 4\pi r^3} \varepsilon^{ijk} \omega_j r^l \left(\frac{4\pi a^2}{3} \delta_{kl} \right)$$

$$A^i = \underbrace{\frac{1}{3}\frac{a^4 \sigma}{c}}_{\frac{(Qa^2/3c)}{4\pi}} \underbrace{\varepsilon^{ijk} \omega_j r_k}_{(\vec{\omega} \times \vec{r})^i} \Big/ r^3$$

$$\vec{A} = \frac{1}{4\pi} \underbrace{\left(\frac{Q a^2 \vec{\omega}}{3c} \right)}_{\equiv \vec{m}} \times \vec{r} \Big/ r^3$$

$$\vec{A} = \frac{1}{4\pi} \frac{\vec{m} \times \vec{r}}{r^3}$$ outside sphere

A similar analysis inside sphere shows

$$\vec{A} = \frac{1}{4\pi}\left(\frac{Q}{3ac}\right)\vec{\omega} \times \vec{r}$$

So

$$\vec{A} = \frac{1}{4\pi}\frac{\vec{m} \times \vec{r}}{r^3} \quad \text{outside}$$

$$\vec{A} = \frac{1}{4\pi}\left(\frac{\vec{m}}{a^3}\right) \times \vec{r} \quad \text{inside}$$

$$\vec{B} = \nabla \times \vec{A} = \begin{cases} \dfrac{1}{4\pi r^3}\left[3\hat{r}(\hat{r}\cdot m) - \vec{m}\right] & \text{outside} \\[2ex] \dfrac{1}{4\pi}\left(\dfrac{2\vec{m}}{a^3}\right) \leftarrow \text{constant} & \text{inside} \end{cases}$$

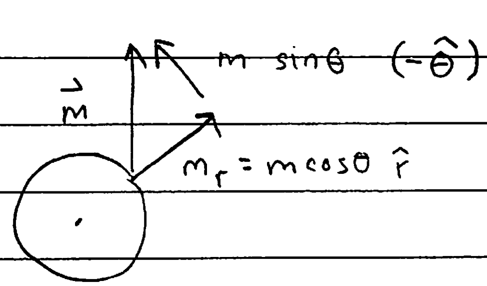
Check Boundary conditions: $\perp$ to surface

$$B_r^{out} - B_r^{in} = 0 \qquad m_r = m\cos\theta = \vec{m}\cdot\hat{r}$$

$$B_r^{out} = \underbrace{\frac{1}{4\pi a^3}\left(3m\cos\theta - m\cos\theta\right)}_{\text{out}} = \underbrace{\frac{1}{4\pi a^3}\frac{(2m\cos\theta)}{a^3}}_{\text{in}} = B_r^{in}$$

Rotating Sphere pg. 6 - Checking BC

Similarly



In $\hat{\theta}$ direction

$$B^{\theta}_{out} = \frac{1}{4\pi a^3} \left(+ m \sin\theta \right)$$

$$B^{\theta}_{in} = \frac{1}{4\pi a^3} 2 \left(- m \sin\theta \right)$$

So

$$B^{\theta}_{out} - B^{\theta}_{in} = \frac{1}{4\pi a^3} 3 m \sin\theta$$

$$m = \frac{Q a^2}{3c} \omega$$

$$= \underbrace{\sigma \omega a \sin\theta}_{\text{Surface current } \sigma v}$$

$$\sigma = \frac{Q}{4\pi a^2}$$

$$B^{\theta}_{out} - B^{\theta}_{in} = K \checkmark$$

Aside (but very Important)

A note on rotationally invariant integrals

$I^{ij} = \int d\Omega \, x^i x^j$ ← we studied this integral two pages back

I^{ij} is a symmetric tensor and must be proportional to available tensors that can be constructed. In this case the only available tensor is δ^{ij}

$$\int d\Omega \, x^i x^j = C \delta^{ij}$$ ← see specific case two pages back

Contracting (setting equal i and j)

$$\int d\Omega \, \underbrace{\overbrace{\vec{x}\cdot\vec{x}}^{x^i x_i}}_{=a^2} = C \underbrace{\delta^i_{\ i}}_{=3}$$

$$4\pi a^2 = C\,3 \implies C = 4\pi a^2/3$$

Now Consider

$$\underbrace{\int d\Omega \, x^i x^j x^k x^l}_{\text{Symmetric in all four indices}} = C \underbrace{\left[\delta^{ij}\delta^{kl} + \delta^{ik}\delta^{jl} + \delta^{il}\delta^{jk}\right]}_{\text{This is the only completely symmet tensor that can be constructed out of } \delta^{ij}}$$

Important Aside pg. 2 - on rotationally invariant integrals

So contracting ij and kl

$$\int d\Omega \, \underbrace{x^i x_i}_{a^2} \, \underbrace{x^k x_k}_{a^2} = C \left[\delta^i{}_i \delta^k{}_k + \delta^{ik} \delta_{ik} + \delta^i{}_k \delta_i{}^k \right]$$

$$= C \left[3 \cdot 3 + 3 + 3 \right]$$

$$4\pi a^4 = C \, 15$$

$$\frac{4\pi a^4}{15} = C$$

Then

$$\int d\Omega \, x^i x^j x^k x^l = \frac{4\pi a^4}{15} \left[\delta^{ij} \delta^{kl} + \delta^{ik} \delta^{jl} + \delta^{il} \delta^{jk} \right]$$

Take specific case; $z = r\cos\theta = a\cos\theta$

$$I^{3333} = \int d\Omega \, z^4$$

$$= \int d\phi \, d(\cos\theta) \, (\cos\theta)^4 \, a^4$$

$$= \frac{2\pi a^4}{5} \cdot 2 = \frac{4\pi a^4}{5} = \frac{4\pi a^4}{15} \overbrace{\left[\delta^{zz}\delta^{zz} + \delta^{zz}\delta^{zz} + \delta^{zz}\delta^{zz} \right]}^{=3}$$