Last Time

- Discussed Magnetostatics

$$\nabla \times B = \vec{j}/c$$

$$\nabla \cdot B = 0$$

For steady current which is specified

$$\nabla \cdot \vec{j} = 0$$

After introducing the vector potential $\vec{B} = \nabla \times \vec{A}$ and the coulomb gauge $\nabla \cdot A = 0$

Find

$$\nabla \times (\nabla \times \vec{A}) = j/c$$

$$-\nabla^2 \vec{A} + \cancel{\nabla(\nabla \cdot A)} = j/c$$

$$\boxed{-\nabla^2 \vec{A} = j/c}$$

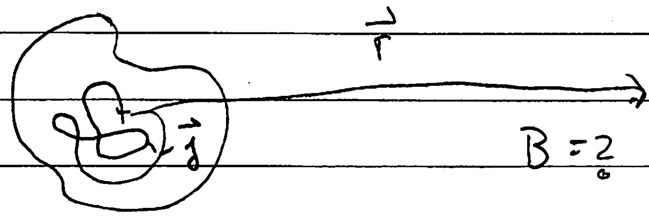
also useful in a few cases

Then

$$\boxed{\vec{A}(\vec{r}) = \int d^3x \, \frac{\vec{j}(x)/c}{4\pi \, |\vec{r} - \vec{x}|}}$$

Can use this to solve problems

Expansion of the magnetic fields at great distances



Start

$$A^i = \frac{1}{c}\int d^3x \, \frac{j^i(x)}{4\pi|\vec{r}-\vec{x}|^{1/2}}$$

$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{(r^2+x^2-2\vec{x}\cdot\vec{r})^{1/2}} \simeq \frac{1}{r} + \frac{\vec{x}\cdot\vec{r}}{r^3} + \ldots$$

Will find: no monopole

$$\boxed{A^i = 0 + \frac{(\vec{m}\times\hat{r})^i}{4\pi r^2}}$$

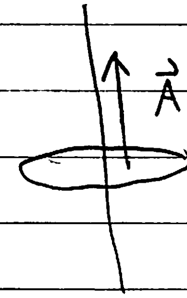
where

$$\boxed{\vec{m} = \frac{1}{2}\int \vec{x} \times \frac{\vec{j}(x)}{c}\, d^3x}$$

is the magnetic moment

Expansion of magnetic fields pg. 2

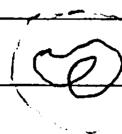
For a current loop in plane



$$d^3x \, \vec{j} = I \, d\vec{l}$$

and

$$\boxed{\vec{m} = \frac{(I/c)}{2}\int \vec{x} \times d\vec{l} = \frac{I}{c} A}$$



← current makes closed loop

Prf

from last page

$$A^i = \cancel{\frac{1}{r}\int \frac{d^3x}{4\pi} \frac{J^i(x)}{c}} + \underbrace{\left[\int d^3x \; J^i x^l\right]}_{I^{il}} \frac{r_l}{4\pi r^3 c}$$

Now Look at :

$$I^{il} = \frac{1}{c}\int d^3x \; J^i x^l$$

$$= \underbrace{\frac{1}{2c}\int d^3x \, (J^i x^l + J^l x^i)}_{I_S^{il}} + \underbrace{\frac{1}{2c}\int (J^i x^l - J^l x^i)}_{I_A^{il}}$$

$$I_S^{il} = \frac{1}{2c}\int d^3x \; J^i x^l + J^l x^i$$

Since $\partial_m J^m = 0$ and $\partial_m x^i = \delta^i_m$

$$= \frac{1}{2c}\int d^3x \; \partial_m (J^m x^i x^l)$$

Since J^m is bounded

$$I_S^{il} = 0$$

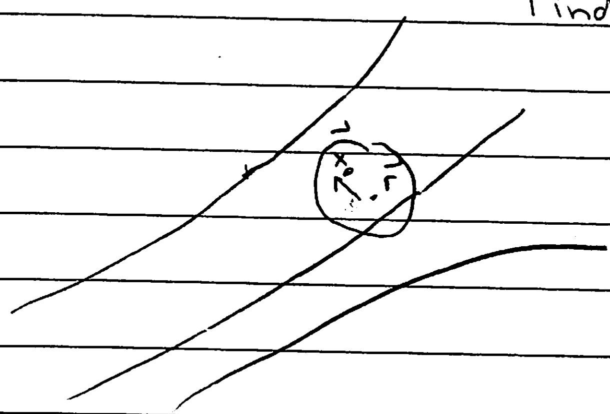
Forces on Current Distributions

- The

$$\vec{F} = \int \frac{\vec{J}(x)}{c} \times \vec{B}(\vec{x})\, d^3x$$

Find: $\boxed{\vec{F} = \vec{\nabla}(\vec{m}\cdot\vec{B}) = -\nabla U}$

magnetic moment



$$\boxed{\vec{U} = -\vec{m}\cdot\vec{B}}$$

Prf

$$F_i = \int d^3x\, \varepsilon_{ijk}\, J^j(x) B^k(x)$$

$$\vec{B}^k(\vec{x}) = B^k(\vec{r}) + \delta x^\ell \partial_\ell B^k(\vec{r}) \qquad \delta x^\ell = x^\ell - r^\ell$$

Plugging in

$$F_i = \underbrace{\left[\int d^3x\, J^j \delta x^\ell\right]}_{\text{Same as before } = -\varepsilon^{j\ell n} m_n} \varepsilon_{ijk}\, \partial_\ell B^k(\vec{r})$$

(see two pages back)

$$F_i = -\varepsilon^{j\ell n} m_n\, \varepsilon_{ijk}\, \partial_\ell B^k(\vec{r})$$

Using

$$-\varepsilon^{jln}\,\varepsilon_{ijk} = \varepsilon^{jln}\,\varepsilon_{jik}$$

$$= \delta^{l}{}_{i}\,\delta^{n}{}_{k} - \delta^{l}{}_{k}\,\delta^{n}{}_{i}$$

We have

$$F_i = m_n \partial_l B^k \left[\delta^{l}{}_{i}\,\delta^{n}{}_{k} - \delta^{l}{}_{k}\,\delta^{n}{}_{i}\right]$$

$$= m_k \partial_i B^k - \underbrace{m_i \partial_k B^k}_{\nabla \cdot B = 0}$$

$$F_i = \partial_i (\vec{m} \cdot \vec{B}) \checkmark$$

as claimed

From this we see that the potential energy of a magnetic dipole is

$$U = -\vec{m} \cdot \vec{B}$$

Finally we may take the derivative with respect to theta, (or go through another tortuous derivation), to show along the same lines that

$$\vec{\tau} = \vec{m} \times \vec{B}$$

i.e, that the magnetic moment tends to align itsself @ the field