Steady Currents in Matter - Ohms Law

- How to calculate?
- What?? I don't calculate currents I calculate fields. . . Right! You specify a constituent relation and solve for fields.

For an ohmic conductor:

T-odd = dissipative

$$\vec{j} = \sigma \vec{E} + \varkappa_e \partial_t \vec{E} + \ldots$$

for an insulating dielectric we dropped this term, since by definition no current flows for a constant field.

For a conductor we keep this and drop the first gradient which vanishes for constant fields, and is anyway smaller

$$\vec{j} = \sigma \vec{E}$$

$$[j] = \frac{q}{m^2 s} \qquad [E] = \frac{q}{m^2} \qquad [\sigma] = \frac{1}{s}$$

Steady Currents in Matter

Now:

$$\nabla \cdot \vec{j} = 0$$

So

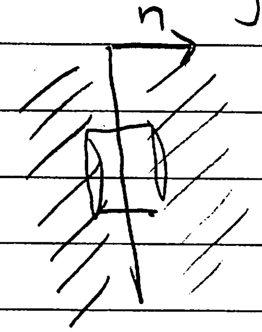
$$\vec{\nabla} \cdot (\sigma \vec{E}) = 0$$

$$\nabla \times \vec{E} = 0$$

Thus we find the Eqn to solve:

$$\boxed{-\vec{\nabla} \cdot (\sigma \nabla \varphi) = 0} \Longrightarrow \underbrace{-\sigma \nabla^2 \varphi = 0}_{\text{for } \sigma \text{ const.}}$$

We need boundary conditions

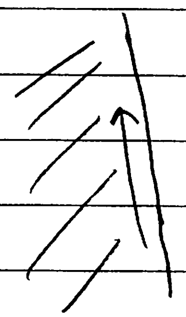
σ_1  σ_2

from $\nabla \cdot \vec{j} = 0$

$$\vec{n} \cdot (\vec{j}_2 - \vec{j}_1) = 0$$

or $\boxed{\sigma_2 E_2^{\perp} = \sigma_1 E_1^{\perp}}$

This is most often used at an ohmic/insulator interface

σ_1  $\sigma_2 = 0$ $\Rightarrow$ Find $\boxed{E_1^{\perp} = 0}$

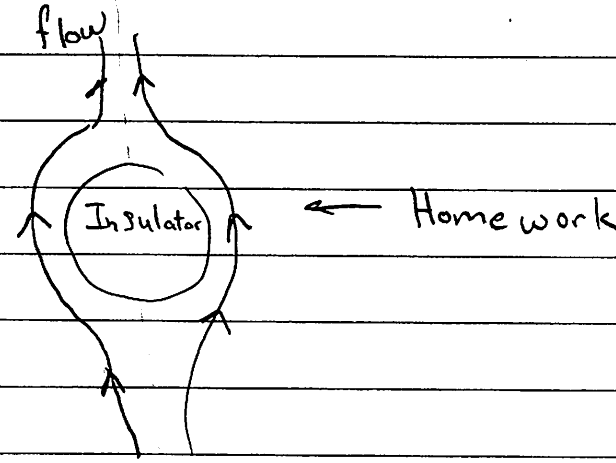
i.e. the electric field and current are parallel to the surface

Notice that the boundary conditions are rather different from the Dirichlet boundary conditions ($\varphi = 0$) we are

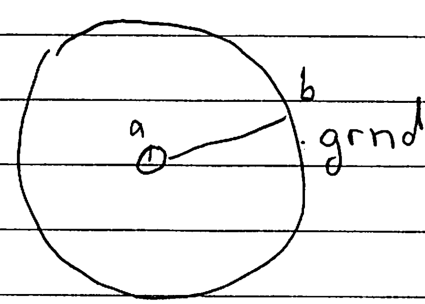
instead specifying the normal derivatives

$$E_1^{\perp} = -\vec{n} \cdot \nabla \varphi = 0$$

This is known as Neumann boundary conditions. The solutions can be rather different, and have a strong analogy with fluid flow



Ex: An electrode injects current I at the origin of an ohmic sheet; the electrode has radius a and the outer rim of the ohmic sheet has radius b. Determine the electric field everywhere, and determine the resistance of the configuration



Problem Solution: We use $\vec{j}$ = current per area

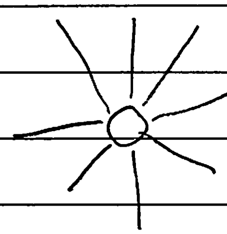
perhaps we should use $\vec{K}$

We want to solve

$$-\sigma \nabla^2 \varphi = 0$$

together with b.c. $\overbrace{-\sigma \frac{\partial \varphi}{\partial n}}^{\sigma E} = \frac{I}{2\pi a}$

"surface" a line in 2D



So

$$\int_{\cancel{\text{circle of}}\ \text{radius } a} \vec{j} \cdot d\vec{S} = \int a\, d\phi \frac{I}{2\pi a}$$

$$= I$$

So solving

$$\frac{1}{\rho} \frac{\partial}{\partial \rho} \rho \frac{\partial}{\partial \rho} \varphi = 0$$

$$\varphi = A + B \ln \rho$$

With B.C. $\varphi\big|_{\rho = b} = 0$ and $-\frac{\partial \varphi}{\partial \rho}\Big|_{\rho = a} = \frac{I}{2\pi \sigma a}$ find

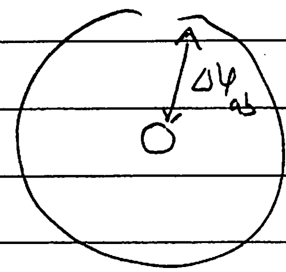
$$\varphi = -\frac{I}{2\pi\sigma} \ln \frac{\rho}{b}$$

$\vec{j} = \sigma \frac{\partial \varphi}{\partial \rho} \hat{\rho} = \frac{I}{2\pi \rho}$ $\Leftarrow$ perhaps we could have gessed this

From Ohms Law

$\Delta\varphi_{ab} = I R$ and our result

$$\Delta\varphi_{ab} = I\left(\frac{-1}{2\pi\sigma}\ln\frac{a}{b}\right)$$

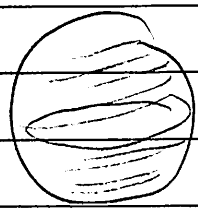


Find $\boxed{R = \frac{1}{2\pi\sigma}\ln\left(\frac{b}{a}\right)}$

Math Discussion

Reduction of Tensor Integrals - A Useful/easy technique

$x = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$



- Three excercises to mastery

(1) $\int d\Omega_x \, x^i x^j = C\delta^{ij}$

$$\underbrace{\int d\Omega_x \, \overbrace{x^i x_i}^{1}}_{4\pi} = C \cdot 3$$

$$\frac{4\pi}{3} = C$$

(2) Consider an integral like this and reduce to scalar:

$$I^i = \int d\Omega \frac{x^i}{1 + \vec{v} \cdot \vec{x}} = \int d\Omega \, x^i f(\vec{x} \cdot \vec{v})$$

Use rotational Symmetry to claim:

$$I^i = A(v)\,\hat{v}^i$$

Now dot both sides with $\hat{v}$

$$I^i \hat{v}_i = A(v) = \int d\Omega \, x^i \hat{v}_i \, f(\vec{x} \cdot \vec{v})$$

So now we are free to take v along z-axis

$$A(v) = \int d\Omega \, \cos\theta \, f(v\cos\theta)$$

$$= 2\pi \int_{-1}^{1} d(\cos\theta) \, \frac{\cos\theta}{1+v\cos\theta}$$

So $I^i = A(v)\, \hat{v}^i$:

③ Consider an integral - Excercise #3

$$I^{ij} = \int d\Omega \, x^i x^j \, f(\vec{x}\cdot\vec{v})$$

Reduce this integral to two scalars:

$$I_1 = \int d\Omega \, \cos^2\theta \, f(v\cos\theta)$$

$$I_2 = \int d\Omega \, f(v\cos\theta)$$

or Better use I_1 +

$$I_3' \equiv \int d\Omega \, \underbrace{\left(\frac{3}{2}\cos^2\theta - \frac{1}{2}\right)}_{P_2(\cos\theta)} f(vc\ldots$$

Solution

$$I^{ij} = C(v)\,\delta^{ij} + D(v)\,\hat{v}^i v^j$$

Better

$$I^{ij} = \frac{1}{3}A(v)\,\delta^{ij} + B(v)\,\underbrace{\left(\hat{v}^i\hat{v}^j - \frac{1}{3}\delta^{ij}\right)}_{\text{Symmetric traceless}}$$

Solution

$$I^{ij} = \frac{1}{3} A(v)\, \delta^{ij} + B(v) \left[\hat{v}^i \hat{v}^j - \frac{1}{3}\delta^{ij} \right]$$

Taking trace

$$I^i{}_i = A(v)$$

$$= \int d\Omega\, \vec{x}\cdot\vec{x}\, f(\vec{x}\cdot\vec{v})$$

$$I_1 = A(v) = \int d\Omega\, f(v\cos\theta)$$

dotting Both sides ⓦ $\hat{v}$

$$\hat{v}_i I^{ij} v_j = \frac{1}{3} A(v) + \frac{2}{3} B(v)$$

$$\int d\Omega\, \vec{x}\cdot\hat{v}\, \vec{x}\cdot\hat{v}\, f(v\cos\theta) = \frac{1}{3} A + \frac{2}{3} B(v)$$

$$I_2 = \frac{1}{3} \overset{I_1}{A} + \frac{2}{3} B$$

$$\frac{3}{2} I_2 - \frac{1}{2} I_1 = B$$

So

$$I^{ij} = \frac{1}{3} I_1(v)\, \delta^{ij} + \overbrace{\left(\frac{3}{2} I_2 - \frac{1}{2} I_1\right)}^{= I_3} \left(\hat{v}^i \hat{v}^j - \frac{1}{3}\delta^{ij}\right)$$

(4)

Constant symmetric traceless tensor

$$T^{ij} = \left[\int_p d^3p \; f(p) \; p^i p^j p^l p^m \right] \chi_{lm}$$

Show that: a scalar integral of f

$$T^{ij} = I \, \chi^{ij}$$

And determine I: <u>Solution</u> start by saying

$$I^{ijlm} = \int d^3p \; f(p) \; p^i p^j p^l p^m = C \left[\delta^{ij}\delta^{lm} + \delta^{il}\delta^{jm} + \delta^{im}\delta^{jl} \right]$$

Contracting all indicéces:

$$I^{i \; l}_{\; i \; l} = C \, [\overbrace{3 \cdot 3 + 3 + 3}^{15}]$$

$$\frac{1}{15} I^{i \; l}_{\; i \; l} = C$$

So we have

$$C = \frac{1}{15} \int \overbrace{d^3p}^{4\pi p^2 dp} \; f(p) \, (p^2)^2$$

$$C = \frac{4\pi}{15} \int_0^\infty dp \; f(p) \, p^6$$

So

$$T^{ij} = C\left[\delta^{ij}\delta^{lm} + \delta^{il}\delta^{jm} + \delta^{im}\delta^{jl}\right] x_{lm}$$

$$= C\left[0 + x^{ij} + x^{ij}\right]$$

$$T^{ij} = 2C\, x^{ij}$$

So

$$\boxed{T^{ij} = \left[\frac{8\pi}{15}\int_0^\infty dp\, f(p)\, p^6\right] x^{ij}}$$