Steady Currents in Matter - Ohm's Law

- How to calculate?

- What?? I don't calculate currents, I calculate fields. Right! You specify a constituent relation and solve for fields.

For an ohmic conductor:

\[ \mathbf{j} = \sigma \mathbf{E} + \kappa \mathbf{\partial}_t \mathbf{E} + \ldots \]

for an insulating dielectric we dropped this term since by definition no current flows for a constant field.

For a conductor we keep this and drop the first gradient which vanishes for constant fields, and is anyway smaller.

\[ \mathbf{j} = \sigma \mathbf{E} \]

\[
\begin{align*}
[j] &= \frac{q}{m^2 s} \\
[E] &= \frac{q}{m^2} \\
[\sigma] &= \frac{1}{s}
\end{align*}
\]
Steady Currents in Matter

Now,

$$\nabla \cdot \vec{j} = 0$$

So

$$\nabla \cdot (\sigma \vec{E}) = 0$$

$$\nabla \times \vec{E} = 0$$

Thus we find the Eqn to solve:

$$- \nabla \cdot (\sigma \nabla \psi) = 0 \implies -\sigma \nabla^2 \psi = 0$$

for $\sigma$ const.

We need boundary conditions

$$\vec{n} \cdot (\vec{j}_2 - \vec{j}_1) = 0$$

or

$$\sigma_2 \vec{E}_1^\perp = \sigma_1 \vec{E}_1^\perp$$

This is most often used at an ohmic/insulator interface

$$\sigma_1 \quad \sigma_2 = 0 \quad \Rightarrow \quad \text{Find } \boxed{\vec{E}_1^\perp = 0}$$

i.e. the electric field and current are parallel to the surface
Notice that the boundary conditions are rather different from the Dirichlet boundary conditions \( \psi = 0 \) we are instead specifying the normal derivatives

\[ E_1 = -\vec{n} \cdot \nabla \psi = 0 \]

This is known as Neumann boundary conditions. The solutions can be rather different, and have a strong analogy with fluid flow.

**Ex:** An electrode injects current \( I \) at the origin of an ohmic sheet, the electrode has radius \( a \) and the outer rim of the ohmic sheet has radius \( b \). Determine the electric field everywhere, and determine the resistance of the configuration.
Problem Solution: We use \( \dot{j} = \text{current per area} \)
perhaps we should use \( \hat{K} \)

We want to solve

\[-\sigma \nabla^2 \Phi = 0 \quad \sigma \in \Omega \]

together with b.c. \(-\sigma \partial_\nu \Phi = I \)
\[\partial_\nu \quad 2\pi a \]

"surface" a line in 2D

\[ \int \dot{j} \cdot dS = \int 2\pi \sigma a \phi \quad I \]
\[ \text{circle of} \quad \text{radius} a \]

So solving

So solving

\[ \int \frac{1}{2} \partial_\rho \Phi = 0 \]
\[ \partial_\rho \quad \partial_\phi \]

\[ \Phi = A + B \ln \rho \]

With b.c. \( \Phi |_{\rho=b} = 0 \) and \(-\partial_\rho \Phi |_{\rho=a} = \frac{I}{2\pi \sigma a} \)

\[ \Phi = -\frac{I}{2\pi \sigma} \ln \rho \quad |_{b} \]

\[ \hat{j} = \sigma \partial_\rho \Phi \quad \hat{\rho} = \frac{I}{2\pi \rho} \]

\[ \hat{j} \leftrightarrow \text{perhaps we could have guessed this} \]
From Ohms Law

\[ \Delta \Phi_{ab} = I R \quad \text{and our result} \]

\[ \Delta \Phi_{ab} = I \left( -\frac{1}{2\pi \sigma} \ln \frac{b}{a} \right) \]

Find \[ R = \frac{1}{2\pi \sigma} \ln \left( \frac{b}{a} \right) \]
Math Discussion

Reduction of Tensor Integrals - A Useful/ Easy Technique

\[ x = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \theta) \]

* Three exercises to mastery

1) \[ \int \Omega x^i x^j = C S^i_j \]

\[ \int \Omega x^i x_i = C \cdot 3 \]

\[ \frac{4\pi}{3} = C \]

2) Consider an integral like this and reduce to scalar:

\[ I^i = \int \Omega x^i = \int \Omega x^i f(x \cdot \hat{v}) \]

Use rotational symmetry to claim:

\[ I^i = A(v) \hat{v}^i \]

Now dot both sides with \( \hat{v} \):

\[ I^i \hat{v}_i = A(v) = \int d\Omega x^i \hat{v}_i f(x \cdot \hat{v}) \]
So now we are free to take \( v \) along \( z \)-axis

\[
A(v) = \int \cos \theta \ f(v \cos \theta) \\
= 2\pi \int_{-1}^{1} d(\cos \theta) \ \frac{\cos \theta}{1 + v \cos \theta}
\]

So, \( J^i = A(v) \ \hat{v}^i \)

3. Consider an integral - Exercise #3

\[
I_{ij}^i = \int d\Omega \ \hat{x}_{i} \hat{x}_{j} \ f(\hat{x} \cdot \hat{v})
\]

Reduce this integral to two scalars:

\[
I_1 = \int d\Omega \ \cos^2 \theta \ f(v \cos \theta) \quad \text{or better use } I_1 + \\
I_2 = \int d\Omega \ f(v \cos \theta) \quad I_3 = \int d\Omega \ \left( \frac{3}{2} \cos^2 \theta - \frac{1}{2} \right) f(v \cos \theta)
\]

Solution

\[
I_{ij}^i = C(v) \ S_{ij}^i + D(v) \ \hat{v}^i \hat{v}^j
\]

Better

\[
I_{ij}^i = \frac{1}{3} A(v) \ S_{ij}^i + B(v) \left( \hat{v}^i \hat{v}^j - \frac{1}{3} \delta_{ij} \right)
\]

Symmetric traceless
Solution

\[ I^i = \frac{1}{3} A(v) \delta^i + B(v) [\hat{v} \cdot \hat{v} - \frac{1}{3} \delta^i] \]

Taking trace

\[ I^i = A(v) \]

\[ = \int d\Omega \, x \cdot x \, f(x \cdot \hat{v}) \]

\[ I_1 = A(v) = \int d\Omega \, f(v \cos \theta) \]

Dotting both sides \( \Omega \) \( \hat{v} \)

\[ \hat{v} \cdot I^i \cdot v = \frac{1}{3} A(v) + 2B(v) \]

\[ \int d\Omega \, \hat{x} \cdot \hat{v} \cdot \hat{x} \cdot \hat{v} \cdot f(v \cos \theta) = \frac{1}{3} A + 2B(v) \]

\[ \frac{1}{3} I_1 \]

\[ I_2 = \frac{1}{3} A + 2B \]

\[ \frac{3}{2} I_2 - I_1 = B \]

So

\[ I^i = \frac{1}{3} \frac{1}{3} A(v) \delta^i + \left( \frac{3}{2} I_2 - I_1 \right) \left( \frac{\hat{v} \cdot \hat{v} - }{3} \right) \]

\[ = I_3 \]
\[ T_{ij} = \int \frac{d^3 p \, f(p) \, p_i p_j p^k p^m}{V} \times \delta_{lm} \] 

Show that: traceless tensor

\[ T_{ij} = t \chi_{ij} \]

And determine \( t \): Solution start by saying

\[ I_{ij} = \int d^3 p \, f(p) \, p_i p_j p^k p^m = C \left[ \delta_{ik} \delta_{jm} + \delta_{ij} \delta_{km} + \delta_{im} \delta_{jk} \right] \]

Contracting all indices:

\[ I_{ij} = C \left[ 3 \cdot 3 + 3 + 3 \right] \]

\[ \frac{1}{15} I_{ij} = C \]

So we have

\[ \frac{1}{15} \int d^3 p \, f(p) \, (p^2)^2 \]

\[ C = \frac{1}{15} \int d^3 p \, f(p) \, (p^2)^2 \]

\[ C = \frac{4}{15} \int_0^\infty dp \, f(p) \, p^6 \]
So

\[ T^{ij} = C \left[ \delta^i_j \delta^m_n + \delta^i_m \delta^j_n + \delta^i_n \delta^j_m \right] x^m \]

\[ = C \left[ 0 + x^i \delta^j_n + x^i \delta^j_m \right] \]

\[ T^{ij} = 2C x^i x^j \]

So

\[ T^{ij} = \left( \frac{8 \pi}{15} \right) \int_0^\infty dp f(p) p^6 x^{i \delta} \]