Electrostatics

Fundamental Eqs

\[ \nabla \cdot \mathbf{E} = \rho \quad \text{and} \quad \mathbf{F} = q \mathbf{E} \]

\[ \nabla \times \mathbf{E} = 0 \]

Since \( \nabla \times \mathbf{E} = 0 \) we can write it as a gradient of a scalar function

\[ \mathbf{E} = -\nabla \varphi \quad \Rightarrow \quad \varphi(x_b) - \varphi(x_a) = -\int_a^b \mathbf{E} \cdot d\mathbf{l} \]

Then substituting this into fundamental Eqs,

\[ -\nabla^2 \varphi = \rho \]

and when \( \rho = 0 \)

Poisson eqn

\[ -\nabla^2 \varphi = 0 \quad \Rightarrow \quad \text{Laplace eqn} \]
Assumed Knowledge (pg. 1)

1. Gauss Law follows from $\nabla \cdot E = \rho$
\[
\int_\mathcal{S} \mathbf{E} \cdot d\mathbf{A} = \oint_\mathcal{C} \mathbf{E} \cdot d\mathbf{l} = \Phi \nabla \cdot \mathbf{E} = \rho
\]

2. Potential Energy
\[
U_E = \frac{1}{2} \int \int \frac{\rho(r) \rho(r')}{4\pi \epsilon_0 |r-r'|} \, dV
= \frac{1}{2} \int \rho(r) \varphi(r) \, dV
\]
with
\[
\varphi(r) = \int \frac{\rho(r')}{4\pi \epsilon_0 |r-r'|} \, dV = \text{the Coulomb potential from all charges}
\]

3. The energy density
\[
\epsilon_E = \frac{1}{2} E^2
\]

4. Forces and Stress Tensor (will discuss below)
\[
\mathbf{F} = \int \rho \mathbf{E} \, dV = -\int dS \mathbf{n} \cdot \mathbf{T}^i_j
\]
\[
\text{Forces on a body}
\]
\[
\mathbf{T}^i_j = (-\epsilon_0 \mathbf{E} \delta^i_j + \delta^i_j \mathbf{E}^2)
\]
\[
\text{Electric stress tensor}
\]
\[
\mathbf{E}^2
\]
\[
\text{Electrostatic stress tensor}
\]
Assumed Knowledge pg. 2

5) Multipole Expansion: for $r \gg x$

$$\psi(\hat{r}) = \frac{1}{4\pi} \int \frac{\rho(x) \, d^3x}{|\hat{r} - \hat{x}|}$$

Center of charge

$$\begin{align*}
\frac{1}{|\hat{r} - \hat{x}|} &= \frac{1}{(r^2 + x^2 - 2 \hat{r} \cdot \hat{x})^{1/2}} \\
&= \frac{1}{r} \left(1 - 2 \frac{\hat{r} \cdot \hat{x}}{r} + \frac{x^2}{r^2} + \frac{3}{2} \frac{(\hat{r} \cdot \hat{x})^2}{r^4} - \frac{1}{2} \frac{x^2}{r^2} + \ldots\right) \\
&= \frac{1}{r} + \frac{\hat{r} \cdot \hat{x}}{r^2} + \frac{\hat{r} \cdot \hat{x}^2}{r^3} \left(3 \frac{x^3}{2} - \frac{x^2 \delta_{ij}}{2}\right) + \ldots
\end{align*}$$

So

$$\phi(r) = \frac{1}{4\pi} \int \left[ \frac{\rho}{r} + \frac{\rho \cdot \hat{r}}{r^2} + \Theta_{ij} \frac{\hat{r} \cdot \hat{r}^3}{r^3} + \ldots \right]$$
Where

\[ Q_V = \int_V \rho(x) \, dV \quad \text{charge (monopole moment)} \]

\[ \vec{p} = \int_V \rho(x) \vec{x} \, dV \quad \text{dipole moment} \]

\[ \Theta_{1y} = \frac{1}{2} \int_V \rho(x) \left( 3x^1 x^1 - x^2 S^1 y \right) \, dV \quad \text{quadrupole moment} \]
Assumed Knowledge pg. 3

6. Assumed knowledge about dipoles

\[ V_\varepsilon = -\rho \cdot E \quad \text{energy} \]

\[ \vec{N} = \vec{\rho} \times \vec{E} \quad \text{torque} \]

\[ \vec{F} = \nabla (\vec{\rho} \cdot \vec{E}) \quad \text{force on dipole} \]
Discussion of Stress Tensor

Solid or fluid

Take an element of solid and ask how the momentum per volume \( \frac{\partial \vec{g}}{\partial t} \) changes. We would expect a conservation law if total momentum is conserved.

\[
\frac{\partial \vec{g}}{\partial t} + \partial_j (\vec{T} \delta^j) = 0 \Rightarrow \frac{\partial \vec{g}}{\partial t} = -\partial_j \vec{T} \delta^j
\]

If \( \vec{g} \) obeys an equation like this then

\[
\frac{\partial P^i}{\partial t} + \frac{1}{2} \int d^3 \gamma \, g^i = -\int dS n_j \, T^i \delta^j = 0
\]

\( T^i \delta^j \) = Force per area

\( = \) Force in \( j \)th direction due to area in \( i \)th direction

if the volume is taken to infinity, this surface integral is irrelevant and total momentum is conserved.
Example: for a gas (at rest) the stress is

\[ T_{ij} = p \delta_{ij} \]

\[ p \text{ pressure } = \text{ force per area} \]

Then:

\[ \partial_i g \delta_{ij} = -\partial_i T_{ij} = -\partial_i p \]

\[ \text{force negative gradient per volume of pressure} \]

Now back to electrodynamics

\[ f \delta = \rho E \delta = \text{force per volume} \]

Use \( \rho = \partial_i E^i \) and \( \nabla \times E = 0 \) \( \partial_i E_j - \partial_j E_i = 0 \)

\[ f \delta = (\partial_i E^i) E \delta \]

\[ \text{homework sec. 3.7} \]

\[ = \partial_i (E^j E^i - \frac{1}{2} \delta_{ij} E^2) \]

So the Stress Tensor \( f \delta = -\partial_i T_{ij} \) is

\[ T_{ij} = -E^i E^j + \frac{1}{2} \delta_{ij} E^2 \]
Solving the Laplace and Poisson Eqs:

- Boundary conditions are needed to solve

1. Example: the potential is specified on the surface of a sphere, \( \Phi(x) \) and vanishes as \( r \to \infty \)

2. \( \Phi(\theta, \phi) \)

3. \( \Phi = ? \)

Once the BC are specified we can solve

2. Other conditions. At a surface

\[ \int E \cdot da = Q_{\text{enc}} \]

\[ (E_2 - E_1) = \sigma \]

\[ E'' - E_1'' = 0 \]

The electric field can be discontinuous across a conductor, but

3. For a conductor, \( \Phi = \text{const} \) \( \Phi \) is constant (otherwise, the gradient of \( \Phi \) would cause the current to flow.)
Expansion in Eigen-funs

- For a complete set of eigen-funs

\[ \sum_n \langle n | \phi \rangle = \delta_{\phi, \phi'} \]

(completeness)

We can expand \( F(\phi) \) in these eigen-funs

\[ \langle \psi | F \rangle = \sum_n \langle x | n \rangle \langle n | \psi \rangle \]

provided \( F(\phi) \) satisfies the same boundary conditions as the eigen-funs.

Ex: If \( F(\phi) \) is \( 2\pi \) periodic

\[ F(\phi) = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} F_m \]

\[ F_m = \frac{1}{2\pi} \int_{0}^{2\pi} e^{-im\phi} F(\phi) \, d\phi \]

The completeness reads

\[ \sum | n \rangle \langle n | = \frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im\phi} e^{-im\phi'} = \sum_{n} \delta(\phi - \phi' + 2\pi n) \]

(acts like identity)

See handout!
1 Series of functions

In each case we are expanding a function in a complete set of eigen-functions

$$\langle x|F \rangle = \sum_n \langle x|n \rangle \langle n|F \rangle$$

(1)

We require that the functions are complete (in the space of functions which satisfy the same boundary conditions as $F$) and orthogonal

$$\sum_n \langle n|n \rangle = I \quad \langle n_1|n_2 \rangle = \delta_{n_1n_2}$$

(2)

In what follows we show the eigen-function in square brackets

(a) A $2\pi$ periodic function $F(\phi)$ is expandable

$$F(\phi) = \sum_{m=-\infty}^{\infty} \left[ e^{im\phi} \right] F_m$$

(3)

$$F_m = \int_0^{2\pi} \frac{d\phi}{2\pi} \left[ e^{-im\phi} \right] F(\phi)$$

(4)

$$\int_0^{2\pi} d\phi \left[ e^{-im\phi} \right] \left[ e^{im'\phi} \right] = 2\pi \delta_{mm'}$$

(5)

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \sum_n \delta(\phi - \phi' + 2\pi n)$$

(6)

(b) A square integrable function in one dimension has a fourier transform

$$F(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[ e^{ikz} \right] F(k)$$

(7)

$$F(k) = \int_{-\infty}^{\infty} dz \left[ e^{-ikz} \right] F(z)$$

(8)

$$\int_{-\infty}^{\infty} dz \; e^{-i(z-k)z'} = 2\pi \delta(k-k')$$

(9)

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} \; e^{ik(z-z')} = \delta(z-z')$$

(10)

(c) A regular function on the sphere $(\theta, \phi)$ can be expanded in spherical harmonics

$$F(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ Y_{\ell m} (\theta, \phi) \right] F_{\ell m}$$

(11)

$$F_{\ell m} = \int d\Omega \left[ Y^{*}_{\ell m} (\theta, \phi) \right] F(\theta, \phi)$$

(12)

$$\int d\Omega \left[ Y^{*}_{\ell m} (\theta, \phi) \right] \left[ Y_{\ell' m'} (\theta, \phi) \right] = \delta_{\ell\ell'} \delta_{mm'}$$

(13)

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \left[ Y_{\ell m} (\theta, \phi) \right] \left[ Y^{*}_{\ell m} (\theta', \phi') \right] = \delta(\cos \theta - \cos \theta') \delta(\phi - \phi')$$

(14)
(d) A function, \( F(\rho) \) on the half line \( \rho = [0, \infty) \), which vanishes like \( \rho^m \) as \( \rho \to 0 \) can be expanded in Bessel functions. This is known as a Hankel transform and arises in cylindrical coordinates

\[
F(\rho) = \int_0^\infty dk \ [J_m(k\rho)] \ F_m(k)
\]  

(15)

\[
F_m(k) = \int_0^\infty \rho d\rho \ [J_m(k\rho)] \ F(\rho)
\]  

(16)

\[
\int_0^\infty \rho d\rho \ [J_m(\rho k)] [J_m(\rho k')] = \frac{1}{k} \delta(k - k')
\]  

(17)

\[
\int_0^\infty kdk \ [J_m(\rho k)] [J_m(\rho' k')] = \frac{1}{\rho} \delta(\rho - \rho')
\]  

(18)