Electrostatics

Fundamental Eqs

$$\nabla \cdot E = \rho \qquad \text{and} \qquad \vec{F} = q\vec{E}$$

$$\nabla \times E = 0$$

Since $\nabla \times \vec{E} = 0$ we can write it as a gradient of a scalar function

$$E = -\nabla \varphi \implies \varphi(x_b) - \varphi(x_a) = -\int_a^b \vec{E} \cdot d\vec{l}$$

Then substituting this into fundamental Eqs.

$$\boxed{-\nabla^2 \varphi = \rho} \qquad \text{and when} \qquad \rho = 0$$

poisson eqn

$$\boxed{-\nabla^2 \varphi = 0} \longrightarrow \text{laplace eqn}$$

Assumed Knowledge (pg.1)

① Gauss Law follows from $\nabla \cdot E = \rho$

$$\int_S \vec{E} \cdot d\vec{S} = Q_V$$

② Potential Energy

$$U_E = \frac{1}{2} \int_r \int_{r'} \frac{\rho(r)\, \rho(r')}{4\pi |\vec{r} - \vec{r}'|}$$

$$= \frac{1}{2} \int_r \rho(r)\, \varphi(r)$$

with

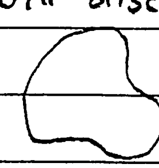
$$\varphi(r) = \int \frac{dr'}{4\pi} \frac{\rho(r')}{|\vec{r} - \vec{r}'|}$$

← the coulomb potential from all charges

③ The energy density

$$u_E = \frac{1}{2} E^2$$

④ Forces and Stress Tensor (will discuss belo



volume integral

$$F^j = \int_V \rho E^j = -\int_{\partial V} dS\, n_i T_E^{ij}$$

force on a body

surface integral of stress tensor

$$T_E^{ij} = \left(-E^i E^j + \frac{1}{2} \delta^{ij} E^2\right)$$

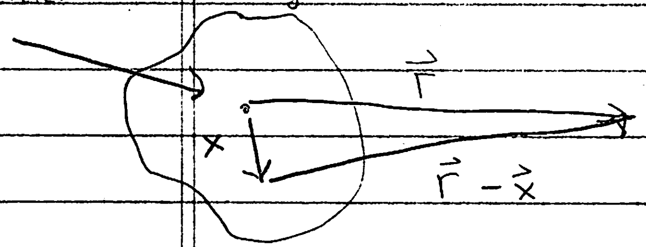
← electrostatic stress tensor

Assumed Knowledge pg. 2

(5) Multipole Expansion: for $r \gg x$

$$\varphi(\vec{r}) = \frac{1}{4\pi}\int \frac{\rho(\vec{x})\, d^3x}{|\vec{r}-\vec{x}|}$$

center of charge



$$\frac{1}{|\vec{r}-\vec{x}|} = \frac{1}{(r^2+x^2-2\vec{r}\cdot\vec{x})^{1/2}} = \frac{1}{r}\left(1 - \frac{2\vec{r}\cdot x}{r^2} + \frac{x^2}{r^2}\right)^{-1/2}$$

$$\simeq \frac{1}{r}\left(1 + \frac{\vec{r}\cdot\vec{x}}{r^2} + \frac{3}{2}\frac{(\hat{r}\cdot x)(r\cdot x)}{r^4} - \frac{1}{2}\frac{x^2}{r^2} + \dots\right)$$

$$\simeq \frac{1}{r} + \frac{\hat{r}^i x^i}{r^2} + \frac{\hat{r}^i \hat{r}^j}{r^3}\left(\frac{3}{2}x^i x^j - \frac{1}{2}x^2\delta^{ij}\right) + \dots$$

So

$$\phi(r) = \frac{1}{4\pi}\left[\frac{Q}{r} + \frac{\vec{p}\cdot\hat{r}}{r^2} + \Theta_{ij}\frac{\hat{r}^i\hat{r}^j}{r^3} + \dots\right]$$

Where

$$Q_V = \int_V \rho(x)$$ ← Charge (monopole moment)

$$\vec{p}^{\,i} = \int_V \rho(x)\,\vec{x}$$ ← dipole moment

$$\Theta^{ij} = \frac{1}{2}\int_V \rho(x)\left(3x^i x^j - \underline{x}^2 \delta^{ij}\right)$$ ← quadrupole moment

Assumed Knowledge pg. 3

(6) Assumed Knowledge about dipoles

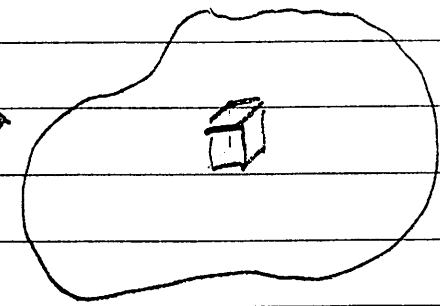
$$V_E = -\vec{p}\cdot\vec{E} \leftarrow \text{energy}$$

$$\vec{N} = \vec{p}\times\vec{E} \leftarrow \text{torque}$$

$$\vec{F} = \nabla(\vec{p}\cdot E) \leftarrow \text{force on dipole}$$

Discussion of Stress Tensor

Solid or fluid



Take an element of solid and ask how the momentum per volume $\equiv \vec{g}_{TOT}$ changes. We would expect a conservation law if total momentum is conserved

$$\frac{\partial g^j_{TOT}}{\partial t} + \partial_i (T^{ij}) = 0 \Rightarrow \frac{dg^j}{dt} = \overbrace{-\partial_i T^{ij}}^{\text{force per volume}}$$

If $\vec{g}$ obeys an equation like this then

$$\frac{\partial P^j_{tot}}{\partial t} = \frac{\partial}{\partial t}\int_V d^3r \; g^j_{TOT} = \underbrace{-\int_{\partial V} dS \, n_i T^{ij}}_{\substack{\text{if the volume is taken to infinity} \\ \text{this surface integral is irrelevant and total} \\ \text{momentum is consv}}} \Rightarrow 0$$

T^{ij} = Force per area

= Force in j^{th} direction due to area in ith direction

Example: for a gas (at rest) the stress is:

$$T^{ij} = p\,\delta^{ij}$$

p ← pressure = force per area

Then:

$$\partial_t g^j = \underbrace{-\partial_i T^{ij}}_{\substack{\text{force}\\\text{per volume}}} = \underbrace{-\partial^j p}_{\substack{\text{negative gradient}\\\text{of pressure}}}$$

Now Back to electrodynamics

$$f^j = \rho E^j = \text{force per volume}$$

Use $\rho = \partial_i E^i$ and $\nabla \times E = 0$ $\quad \partial_i E_j - \partial_j E_i = 0$
to find

$$f^j = (\partial_i E^i)\, E^j$$

homework sec. 3.7

$$= \partial_i \left(E^i E^j - \frac{1}{2} \delta_{ij} E^2 \right)$$

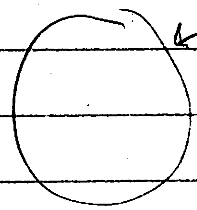
So the Stess Tensor $f^j = -\partial_i T^{ij}$ is

$$\boxed{T^{ij} = -E^i E^j + \frac{1}{2} \delta^{ij} E^2}$$

Solving The Laplace ~~and~~ Poisson Eqs:

- Boundary conditions are needed to solve

(1) Example: the potential is specified on the surface of a sphere, $\Phi_0(\theta, \varphi)$ and vanishes as $r \to \infty$



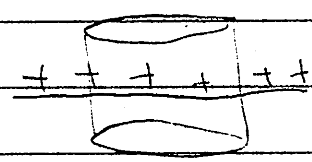
$\Phi_0(\theta, \varphi)$

$\Phi = ?$

Once the BC. are specified we can solve

(2) Other conditions. At a surface

$$\int \vec{E} \cdot da = Q_{enc}$$



$$\boxed{\begin{aligned} (E_2^{\perp} - E_1^{\perp}) &= \sigma \\ E_2^{\parallel} - E_1^{\parallel} &= 0 \end{aligned}}$$

The electric field can be discontinuous across a conductor, but ϕ is con

(3) For a conductor $\phi = \text{const}$ (otherwise the gradient of the conductor would cause the current to flow.)

Expansion in Eigen-fcns:

- For a complete set of eigen-fcns

$$\underbrace{\sum_n |n\rangle\langle n| = \mathbb{1}}_{\text{completeness}}$$

we can expand $F(x)$ in these eigenfcns

$$\langle x|F\rangle = \sum_n \langle x|n\rangle \overbrace{\langle n|F\rangle}^{F(n)}$$

provided $F(x)$ satifies the same boundary conditions as the eigen-fcns

Ex: If $F(\phi)$ is 2π periodic

$$F(\phi) = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im\phi} F_m$$

$$F_m = \int_0^{2\pi} e^{-im\phi} F(\phi)\, d\phi$$

The completeness reads

$$\sum_n |n\rangle\langle n| \Rightarrow \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\phi)} e^{-im\phi'} = \underbrace{\sum_n \delta(\phi-\phi'+2\pi n)}_{\substack{\text{acts like identity} \\ \text{in space of periodic fcn:}}}$$

See handout!

1 Series of functions

In each case we are expanding a function in a complete set of eigen-functions

$$\langle x|F\rangle = \sum_n \langle x|n\rangle \langle n|F\rangle \tag{1}$$

We require that the fucntions are complete (in the space of functions which satisfy the same boundary conditions as F) and orthogonal

$$\sum_n |n\rangle \langle n| = I \qquad \langle n_1|n_2\rangle = \delta_{n_1 n_2} \tag{2}$$

In what follows we show the eigen-function in square brackets

(a) A 2π periodic function $F(\phi)$ is expandable

$$F(\phi) = \sum_{m=-\infty}^{\infty} \left[e^{im\phi}\right] F_m \tag{3}$$

$$F_m = \int_0^{2\pi} \frac{d\phi}{2\pi} \left[e^{-im\phi}\right] F(\phi) \tag{4}$$

$$\int_0^{2\pi} d\phi \left[e^{-im\phi}\right] \left[e^{im'\phi}\right] = 2\pi\delta_{mm'} \tag{5}$$

$$\frac{1}{2\pi} \sum_{m=-\infty}^{\infty} e^{im(\phi-\phi')} = \sum_n \delta(\phi - \phi' + 2\pi n) \tag{6}$$

(b) A square interable function in one dimension has a fourier transform

$$F(z) = \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left[e^{ikz}\right] F(k) \tag{7}$$

$$F(k) = \int_{-\infty}^{\infty} dz \left[e^{-ikz}\right] F(z) \tag{8}$$

$$\int_{-\infty}^{\infty} dz\, e^{-iz(k-k')} = 2\pi\delta(k-k') \tag{9}$$

$$\int_{-\infty}^{\infty} \frac{dk}{2\pi} e^{ik(z-z')} = \delta(z-z') \tag{10}$$

(c) A regular function on the sphere (θ, ϕ) can be expanded in spherical harmonics

$$F(\theta,\phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [Y_{\ell m}(\theta,\phi)]\, F_{\ell m} \tag{11}$$

$$F_{\ell m} = \int d\Omega\, [Y^*_{\ell m}(\theta,\phi)]\, F(\theta,\phi) \tag{12}$$

$$\int d\Omega\, [Y^*_{\ell m}(\theta,\phi)]\, [Y_{\ell' m'}(\theta,\phi)] = \delta_{\ell\ell'}\delta_{mm'} \tag{13}$$

$$\sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} [Y_{\ell m}(\theta,\phi)]\, [Y^*_{\ell m}(\theta',\phi')] = \delta(\cos\theta - \cos\theta')\delta(\phi-\phi') \tag{14}$$

(d) A function, $F(\rho)$ on the half line $\rho = [0, \infty]$, which vanishes like ρ^m as $\rho \to 0$ can be expanded in Bessel functions. This is known as a Hankel transform and arises in cylindrical coordinates

$$F(\rho) = \int_0^\infty kdk \; [J_m(k\rho)] \; F_m(k) \tag{15}$$

$$F_m(k) = \int_0^\infty \rho d\rho \; [J_m(k\rho)] \; F(\rho) \tag{16}$$

$$\int_0^\infty \rho d\rho \; [J_m(\rho k)] \, [J_m(\rho k')] = \frac{1}{k}\delta(k - k') \tag{17}$$

$$\int_0^\infty kdk \; [J_m(\rho k)] \, [J_m(\rho' k)] = \frac{1}{\rho}\delta(\rho - \rho') \tag{18}$$