Last Time:

Started to talk about Faraday's Law

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \frac{j}{c} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B} \implies \underbrace{\int \vec{E}\cdot d\vec{l} = \frac{1}{c}\int \partial_t \vec{B}\cdot d\vec{a}}_{\text{Induced Emf} \,\propto\, \text{change in magnetic flux}}$$

Then we talked about the $1/c$ expansion:

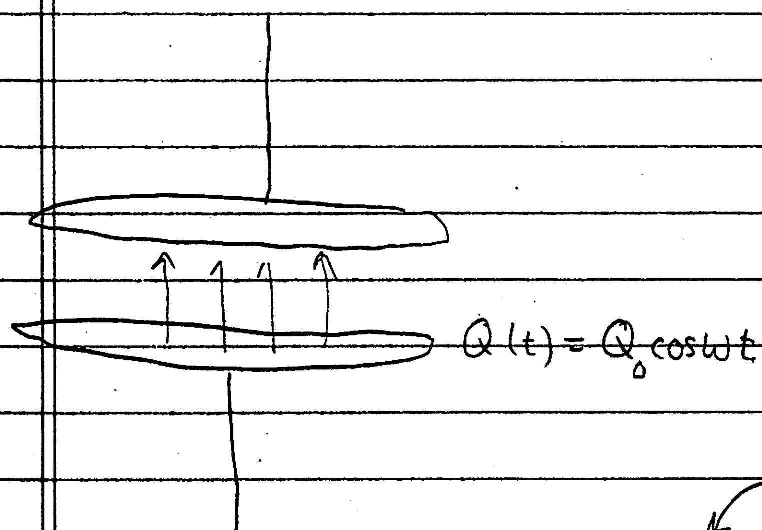
$$\left.\begin{aligned} \nabla \cdot E^{(0)} &= \rho \\ \nabla \times E^{(0)} &= 0 \end{aligned}\right\} \text{Electro Statics}$$

$$\left.\begin{aligned} \nabla \times B^{(1)} &= j + \frac{1}{c}\partial_t E^{(0)} \\ \nabla \cdot B &= 0 \end{aligned}\right\} \text{Magnetostatics + Displacement current}$$

$$\underbrace{\nabla \times E^{(2)}}_{E_{\text{induced}}} = -\frac{1}{c}\partial_t B^{(1)}$$

Then you can continue and solve for $B^{(3)}$:

We worked through an important example



induced emf by changing B-field

$$E^z = \frac{Q_0 \cos\omega t}{\pi R^2}\left[1 - \frac{1}{4}\left(\frac{\omega\rho}{c}\right)^2 + \dots\right]$$

$$B^\phi \simeq - \frac{Q \sin\omega t}{\pi R^2}\left[\frac{\omega\rho}{2c} + O\left(\left(\frac{\omega R}{c}\right)^3\right) + \dots\right]$$

caused by displacement current.

- Calculated the energy in the fields. Found that the electric energy is reduced at high frequency, and the magnetic energy is increased

- $\vec{B}$ invoves odd powers (since it is T-odd)
  $\vec{E}$ " even powers (since it is T-even)
  in $(\omega R/c)$

Gauge Potentials and the Quasi-Static approx

Now:

$$\nabla \cdot E = \rho$$

$$\nabla \times B = \bar{j} + \frac{1}{c}\partial_t E$$

$$\nabla \cdot B = 0$$

$$\nabla \times E = -\frac{1}{c}\partial_t \vec{B}$$

Since $\nabla \cdot B = 0$ we introduce $\boxed{B = \nabla \times A}$ finding

$$\nabla \times E = -\frac{1}{c}\partial_t (\nabla \times \vec{A})$$

Or

$$\nabla \times \left(\vec{E} + \frac{1}{c}\partial_t \vec{A}\right) = 0$$

Thus since the curl of $\vec{E} + \partial_t \vec{A}$ is zero, we write it as the gradient of a scalar fcn $-\nabla \varphi$

$$\vec{E} + \frac{1}{c}\partial_t \vec{A} = -\nabla \varphi$$

$$\boxed{\vec{E} = -\frac{1}{c}\partial_t \vec{A} - \nabla \varphi}$$

Now substituting into $\nabla \times B$ eqn:

$$\underbrace{-\nabla^2 \vec{A} + \vec{\nabla}(\nabla \cdot \vec{A})}_{\nabla \times (\nabla \times \vec{A})} = \frac{\vec{J}}{c} + \frac{1}{c}\partial_t \overbrace{\left(-\frac{1}{c}\partial_t \vec{A} - \nabla \varphi\right)}^{= E}$$

Regrouping:

$$-\underbrace{\left(-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} + \nabla^2\right)}\vec{A} + \vec{\nabla}\left(\frac{1}{c}\partial_t \varphi + \nabla \cdot \vec{A}\right) = \frac{\vec{J}}{c}$$

$$\Box \equiv -\frac{1}{c^2}\partial_t^2 + \nabla^2 = \text{D'alembertian, wave operator}$$

So Find that the eqn for "$\vec{A}$"

$$\boxed{-\Box \vec{A} + \vec{\nabla}\underbrace{\left(\frac{1}{c}\partial_t \varphi + \vec{\nabla} \cdot \vec{A}\right)}_{\text{gauge piece}} = \vec{J}/c}$$

Then the Electric field eqn:

$$\nabla \cdot E = \rho$$

$$\nabla \cdot \left(-\frac{1}{c}\partial_t \vec{A} - \nabla \varphi\right) = \rho$$

Then

$$\boxed{\underbrace{-\nabla^2\varphi}_{\text{laplace eqn}} \underbrace{- \frac{1}{c}\partial_t \nabla\cdot\vec{A}}_{\text{gauge piece}} = \rho}$$

Two common choices:

(1) Coulomb Gauge $\nabla\cdot\vec{A} = 0$

$$\boxed{\begin{aligned} -\nabla^2\varphi &= \rho \\ -\Box\vec{A} &= \vec{j}/c + \frac{1}{c}\partial_t(-\nabla\varphi) \end{aligned}}$$

- Solve the laplace eqn for φ. This acts like a current, $\partial_t(-\nabla\varphi)$, solve for $\vec{A}$
- Good for quasi-static problems:

(2) Covariant Gauge

Take

covariant gauge

$$\boxed{\frac{1}{c}\frac{\partial \vec{A}}{\partial t} + \nabla \cdot \vec{A} = 0}$$

As the time dep generalization of the Coulomb gauge. Then

$$-\nabla^2 \varphi - \frac{1}{c}\partial_t \overbrace{\nabla \cdot A}^{-\frac{1}{c}\partial_t \varphi} = \rho$$

Becomes

$$-\nabla^2 \varphi + \frac{1}{c^2}\partial_t^2 \varphi = \rho$$

Or

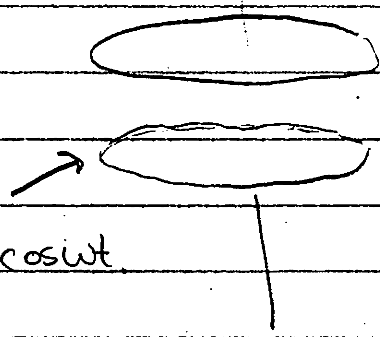
$$\boxed{-\Box \varphi = \rho} \qquad \Box = -\frac{1}{c^2}\partial_t^2 + \nabla^2$$

and

$$\boxed{-\Box \vec{A} = j/c}$$

Important Problem (2nd Time) in Coulomb Gauge:

$$-\nabla^2\varphi = \rho$$



$$\left[-\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\vec{A} = \frac{\vec{J}}{c} + \frac{1}{c}\partial_t(-\nabla\varphi)$$

$Q(t) = Q_0 \cos\omega t$

0th: Then the 0-th order in $1/c$:

$$-\nabla^2\varphi = \cancel{\rho}^{\,0} \qquad \vec{A} = 0$$

Find $\varphi = -\frac{Q(t)}{\pi R^2} z$ ⟸ Actually true to all orders

1st: At first order:

$$\frac{1}{c^2}\cancel{\frac{\partial^2}{\partial t^2}}^{\text{2nd order}} - \nabla^2\vec{A} = \cancel{\frac{\vec{J}}{c}}^{\,0} + \overbrace{\frac{1}{c}\partial_t(-\nabla\varphi)}^{\text{source}}$$

So

$$-\nabla^2\vec{A} = -\frac{Q_0 \sin\omega t}{\pi R^2}\left(\frac{\omega}{c}\right)$$

Then

$$-\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial A^z}{\partial\rho} = -\frac{Q_0\sin\omega t}{\pi}\left(\frac{\omega}{c}\right)$$

So

$$A^z = -\frac{Q_0 \sin \omega t}{\pi R^2}\left(\frac{-\omega \rho^2}{4c}\right) + \text{fcn of } z$$

Then the gauge condition $\nabla \cdot A = 0$ fixes that fcn of z = at most constant

$$B = \vec{\nabla} \times \vec{A}$$

$$B_\phi = -\frac{\partial A^z}{\partial \rho}$$ ← note that B_ϕ indep of fcn of z anyway

$$\boxed{B_\phi = -\frac{Q_0 \sin \omega t}{\pi R^2}\left(\frac{\omega \rho}{2c}\right)}$$

Agrees @ Before

2nd: Note that in the coulomb gauge the second order contribution is very easy to work out.

$$\varphi = -\frac{Q}{\pi R^2} \cos\omega t \, z \qquad \text{is exact}$$

Further $\vec{A}(t,z)$ is the reversal odd so $\vec{A}(t,z)$ must be odd in frequency so can not have second order terms

So $$E = -\frac{1}{c}\partial_t \vec{A} - \nabla\varphi$$

$$\vec{E} = \underbrace{-\frac{1}{c}\partial_t \vec{A}^{(1)}}_{\equiv E^{(2)}} + \vec{E}^{(0)}$$

$$E^{(2)} = -\frac{Q_0 \cos\omega t}{\pi R^2}\left(\left(\frac{\omega\rho}{c}\right)^2 \frac{1}{4}\right)$$

(For the third time)

<u>Important Prob in Cov. Gauge</u>

$$\left[+\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right]\varphi = \cancel{\rho}^{\,0}$$

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\vec{A} = \cancel{\vec{J}/c}^{\,0}$$

0th

$$-\nabla^2\varphi = \rho \implies \boxed{\varphi = -\frac{Q(t)}{\pi R^2}\, z}$$

1st 2nd

$$\cancel{\frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2}} - \nabla^2\vec{A} = 0$$

And gauge condition:

$$\frac{1}{c}\partial_t\varphi + \vec{\nabla}\cdot\vec{A} = 0$$

So find $A^z = \frac{1}{c}\left(\frac{\partial_t Q(t)}{\pi R^2}\right)\frac{z^2}{2} + f(\rho, t)$

Then

$$-\nabla^2\left[f(\rho,t) + \frac{\partial_t Q(t)}{c\pi R^2}\frac{z^2}{2}\right] = 0$$

$$-\nabla^2 f(\rho,t) = \frac{1}{c}\frac{\partial_t Q}{\pi R^2}$$ ← same eqn as before

$$\boxed{A^z = -\frac{Q_0 \sin\omega t}{\pi R^2}\left[\frac{\omega z^2}{2c} - \frac{\omega\rho^2}{4c}\right]}$$

Then

$$\vec{B} = \nabla \times \vec{A}$$

$$\vec{B} = \hat{\phi}\left(-\frac{\partial A^z}{\partial \rho}\right)$$

For $\vec{A}$ which only has z component and is azimuthally symmetric

$$\boxed{\vec{B} = \hat{\phi}\left[\frac{Q_0 \sin \omega t}{\pi R^2} \frac{\omega \rho}{2c}\right]}$$

Same as before note however that $\vec{A}$ differs by a function of z between the covariant and coulomb gauges. But.. this does not affect B

$$B_\phi = -\frac{\partial A^z}{\partial \rho}$$