Last Time

Introduced the equations for the potentials

Coulomb Gauge: $\nabla \cdot \vec{A} = 0$

$$-\nabla^2 \varphi = \rho$$

$$\left[+ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \vec{A} = \frac{\vec{j}}{c} + \frac{1}{c} \partial_t (-\nabla \varphi)$$

Lorentz Gauge: $\frac{1}{c} \frac{\partial \varphi}{\partial t} + \nabla \cdot \vec{A} = 0$

$$\left[+ \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \varphi = \rho$$

$$\left[\frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \nabla^2 \right] \vec{A} = j/c$$

Then

$$B = \nabla \times \vec{A}$$

$$E = -\frac{1}{c} \partial_t \vec{A} - \vec{\nabla} \varphi$$

Quasi-Statics

Described How we can use these equations to solve for the induced fields to each order in $1/c$

Ex Coulomb

$I(t) = I_0 \cos\omega t \; \hat{z}$

0th $\qquad \varphi^{(0)} = 0$

1st $\qquad -\nabla^2 \vec{A} = \frac{I}{c}\,\delta^2(x_\perp)\,\hat{z}$

$$A^z = -\frac{I(t)/c}{2\pi}\ln\rho + C \qquad B = \nabla \times A$$

2nd $\qquad \varphi^{(2)} = 0$

$$A^{(2)} = 0$$

$$\vec{E} = -\frac{1}{c}\partial_t \vec{A}^{(1)} - \cancel{\nabla\varphi}$$

$$\vec{E} = \left[-\frac{1}{c^2}\partial_t I(t)\right]\frac{\ln\rho}{2\pi}\,\hat{z} \Leftarrow$$ Gives induced field automaticall

Quasi-statics in metals: (Intro)

(1) • Free charge rushes to the surface

$$\partial_t \rho + \nabla \cdot \vec{j} = 0$$

Now for a metal

$$\bar{j} = \sigma E$$

$$\partial_t \rho + \underbrace{\nabla \cdot (\sigma E)} = 0$$

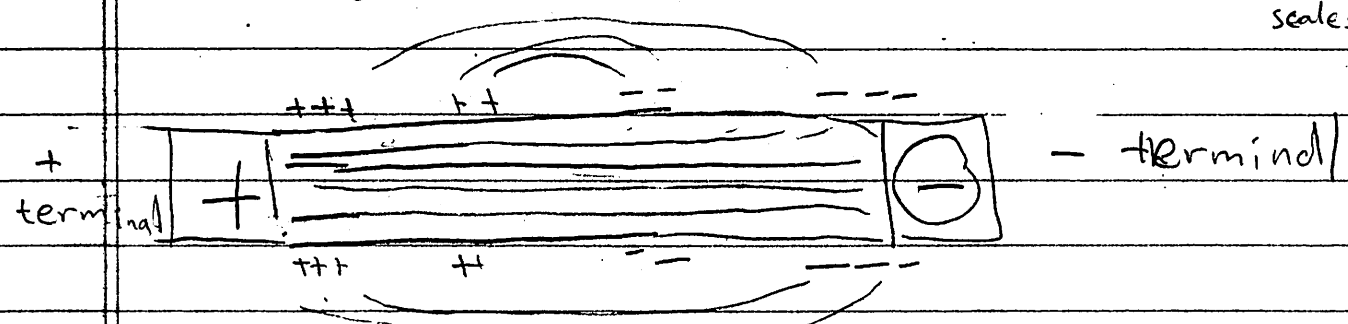
$$\partial_t \rho + \sigma \underbrace{\nabla \cdot E}_{=\rho} = 0$$

$$\partial_t \rho + \sigma \rho = 0$$

$$\rho = \rho_0 e^{-t\sigma}$$

actually this is too fast our macro theory of electrodynam doesn't apply on such time scales;

All charge rushes to the surface:



- The surface charges (and their gradients) cause current to flow down the wire

- They are also responsible for the change in potential from one end of the terminal to the other

How fast is $1/\sigma$?

$$\left.\begin{aligned} \vec{j}_{HL} &= \sigma_{HL} E_{HL} \\ \vec{j}_{mks} &= \sigma_{mks} E_{mks} \end{aligned}\right\} \quad j_{HL} = \frac{j_{mks}}{\sqrt{\varepsilon_0}} \qquad E_{HL} = \sqrt{\varepsilon_0}\, E_{mks}$$

$$\boxed{\sigma_{HL} = \sigma_{mks}/\varepsilon_0}$$

σ_{mks} for Cu $\sim 10^7 \dfrac{1}{\Omega \cdot m}$

Find

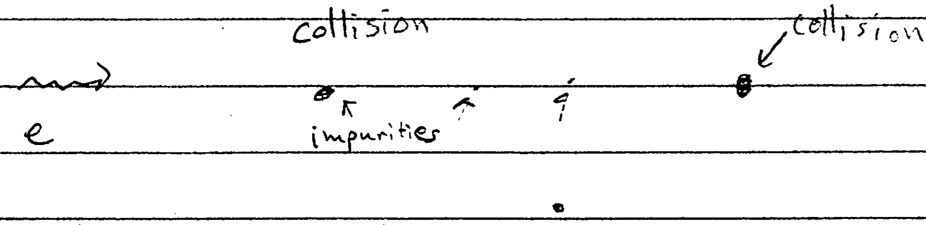
$$\sigma_{HL} \simeq 1.12 \times 10^{18} \frac{1}{s} \quad \frac{\sigma_{mks}}{(10^7\ \Omega m)}$$

Wow! $\sim \dfrac{1\ \mathring{A}}{c}$ Why so fast, Note

$$e\sigma_{HL} = 1.6\times10^{-19}\,C \quad 10^{+18}\,\frac{1}{s} \quad \frac{\sigma_{HL}}{10^{+18}\,1/s}$$

$$\boxed{e\sigma_{HL} = 0.16\ \text{Amps}\left(\frac{\sigma_{HL}}{10^{+18}\ 1/s}\right)}$$

Model



- Electrons are free in metal and occasioncally scatter off impurites.
  Typical momentum is large, $p \sim \hbar/a_0$

$$\lambda_D = \frac{\hbar}{p} \ll l_{mfp}$$

- we are considering time-scales much larger than time between collisions, and much longer than $\frac{\hbar}{E_{kin}} \sim \frac{\hbar}{p^2/2m} \sim \tau_{quant}$ $\quad \tau_{quant} \ll \underset{\substack{\uparrow \\ \text{time between collisions}}}{\tau_c}$

- Classical transport Is good enough:

$$ma = \underbrace{-\frac{mv}{\tau_c}}_{\text{drag force}} + \underbrace{eE}_{\text{driving force}}$$

For steady state $a = 0$

$$v = \frac{eE\tau_c}{m}$$

and

$$\vec{j} = nev$$

$$\vec{j} = \underbrace{\frac{ne^2\tau_c}{m}}_{\equiv \sigma} E$$

So

$$\boxed{\sigma = \frac{ne^2}{m}\tau_c = \omega_p^2 \tau_c}$$

The plasma frequency:

$$\omega_p \equiv \left(\frac{ne^2}{m}\right)^{1/2} \sim \left[\frac{1}{a_0^3}\frac{e^2}{m}\right]^{1/2}$$

$$\sim \left[\left(\frac{e^2}{a_0}\right)\left(\frac{\hbar^2}{m a_0^2}\right)\frac{1}{\hbar^2}\right]^{1/2}$$

$$\sim \left(\frac{27.\,eV}{\hbar}\right) \sim 10^{16} \ \text{rad/s}$$

$\sim$ 1/Time it takes electron to orbit

$\sim$ a typical quantum phase

So for copper $\sigma \sim 10^{18}$ 1/s

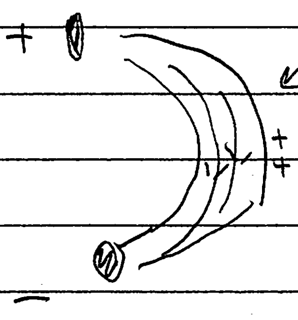
$$\omega_p \tau_c \sim 100 \gg 1$$

and

$$\omega_p (\omega_p \tau_c) \sim 10^{18} \frac{1}{s}$$

$$\tau_c \sim 10^{-14} \frac{1}{s}$$

time it takes to establish current.



at the bend need to have some excess charge to cause the current to be constant

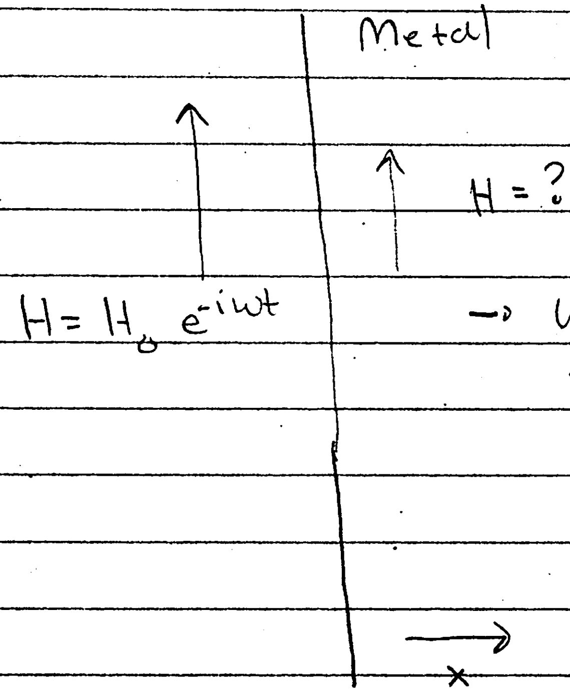
$$\Delta Q \sim \frac{I}{\sigma}$$ from dimensions

$$\Delta Q \sim \frac{1 C/s}{10^{18}/s} \sim 10^{-18} C$$

$\sim$ 10 extra electrons

The Skin Effect & Quasi-statics in metals

Diffusion of magnetic fields



→ We will find that induced currents cause the magnetic fields to decay

Equations for B in metals

If B changes in time then it will induce currents in the metal, which shield the interior

$$\nabla \cdot E = 0$$

$$\nabla \times H = j_{ind}$$

$$\nabla \cdot B = 0$$

$$\nabla \times \vec{E} = -\frac{1}{c}\partial_t \vec{B}$$

Now $j_{ind} = \sigma E$

$$\nabla \times B = \sigma E$$

$$\underbrace{\nabla \times (\nabla \times H)} = \frac{\sigma}{c} \nabla \times E$$

$$\vec{\nabla}(\nabla \cdot H) - \nabla^2 H = -\frac{\sigma}{c^2}\frac{\partial \vec{B}}{\partial t}$$

$B = \mu H$

Or

$$\boxed{+\nabla^2 H = \frac{\mu\sigma}{c^2}\frac{\partial H}{\partial t}}$$

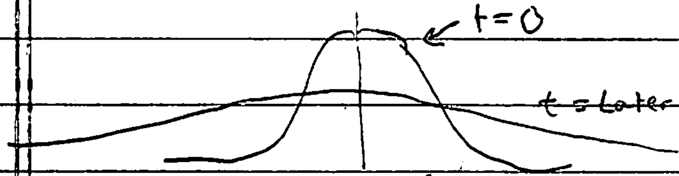
Diffusion Equation

canonical form

$$\boxed{\partial_t n = D \nabla^2 n}$$

A primer on the Diffusion Equation: $\partial_t n = D\nabla^2 n$

A drop of dye in water, The width of drop grows with time



$$(\Delta x)^2 = 2D\,\Delta t$$

- The diffusion equation smears out features.

The Magnetic Diffusion coefficient is:

$$D = \frac{c^2}{\mu\sigma} \Rightarrow \text{units } \left[\frac{c^2}{\sigma}\right] \sim \frac{(\text{meters})^2}{s}$$

$$D \simeq \frac{1\,\text{cm}^2}{\text{millisec}} \quad \text{for } \mu \simeq 1 \quad \text{for copper}$$

Solving the Diffusion Eq: (for problem given two pages ago)

$$H(z,t) = H_0 e^{-i\omega t} h(x)$$

Then $\nabla^2 H = \frac{1}{D}\frac{\partial H}{\partial t}$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{i\omega}{D}\right] h(x) = 0$$

And find trying e^{ikz}:

$$k^2 = \frac{i\omega}{D} \Rightarrow \boxed{k_{\pm} = \pm(1+i)\sqrt{\frac{\omega}{2D}} \equiv \pm\frac{(1+i)}{\delta}}$$

δ is skin depth
see next page for def.

$$\pm\sqrt{i} = \pm\frac{(1+i)}{\sqrt{2}}$$

Solving the Diffusion Eq. pg. 2

Thus we find, selecting $e^{ik_+x} \sim e^{-x/\delta} e^{ix/\delta}$

$$\boxed{H^z = H_0 e^{-x/\delta} \cos(x/\delta - \omega t)} = \mathrm{Re}\, H_0 e^{ik_+x - i\omega t}$$

Find that the magnetic field decays on a length scale:

$$\delta = \sqrt{\frac{2D}{\omega}} = \sqrt{\frac{2c^2}{\omega\mu\sigma}} = \text{(see estimate)}$$

We can calculate the current and electric field

$$\sigma \vec{E} = \vec{j} = \nabla \times B$$

$$\frac{j^y}{c} = -\frac{\partial B^z}{\partial x} = \mathrm{Re}\, -\frac{\partial}{\partial x} H_0 e^{-i\omega t} e^{ik_+x}$$

$$= \mathrm{Re}\, -ik_+ H_0 e^{-i\omega t} e^{ik_+x}$$

$$\boxed{\frac{j^y}{c} = \frac{\sqrt{2}}{\delta} H_0 e^{-x/\delta} \cos(x/\delta - \omega t - \pi/4)}$$

$$-ik_+ = -i\frac{(1+i)}{\delta} = \frac{\sqrt{2}}{\delta} e^{-i\pi/4}$$

(Aside: Estimate of Skin Depth in Copper)

Estimate:

$$\sigma = \sigma_{18} \times 10^{18}\,s$$

$$f = f_{kHz}\, 10^{3}\,Hz = \omega/2\pi \quad \Rightarrow \quad \omega = f_{kHz} \times 2\pi \times 10^{3}\,1/s$$

$$\boxed{\delta = 0.53\ \text{cm}\ \frac{1}{\sqrt{f_{kHz}\,\sigma_{18}}}}$$

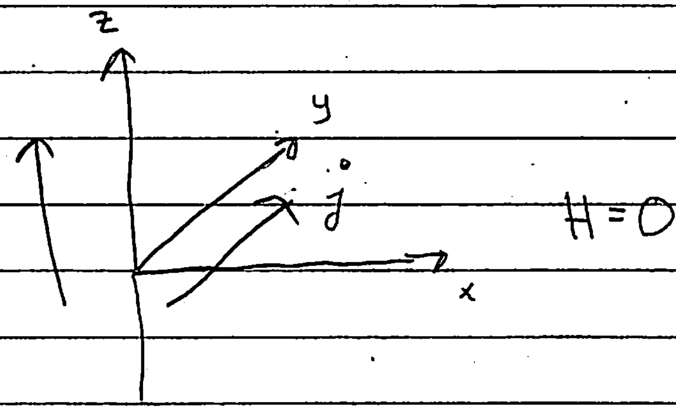
Solving the Diffusion Eq. Pg 3

Picture at $t=0$ and $x=0$

Finite H

But H

decreasing

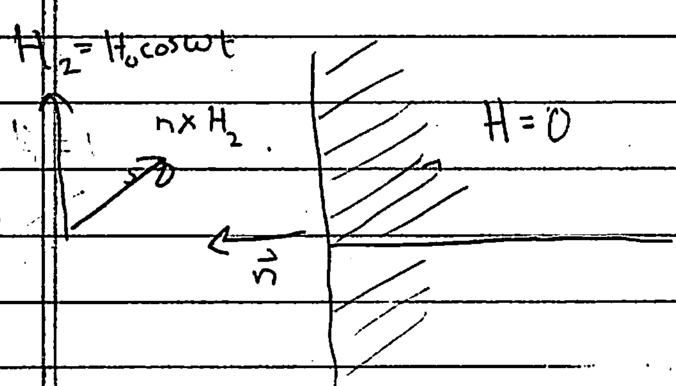


The total current:

$$\frac{K^y}{c} = \int_0^\infty dx \, \frac{\sqrt{2}}{\delta} H_0 e^{-x/\delta} \cos(x/\delta - \omega t - \pi/4)$$

$$\frac{K^y}{c} = H_0 \cos \omega t$$ ← This is what you expect from B.C.

$$n \times (H_2 - \cancel{H_1}^{\,0}) = \frac{K}{c}$$



So

$$n \times H_2 = \frac{K}{c}$$