Last Time

① Finished up quasi-statics, i.e. where c is fast compared to $\frac{L}{T}$, or where $\omega \ll c/L$, by studying quasi-statics in metals

② Metals have their own frequency scales:

- $\frac{1}{\tau_c}$ the frequency of collisions with impurities and each other $\frac{1}{\tau_c} \sim 10^{14}$ Hz
- $\omega_p^2 \simeq \frac{ne^2}{m}$ The plasma frequency. Natural coulomb oscillation/orbital frequency of electrons

$$\sim \frac{13.6\,eV}{\hbar} \sim 10^{16}\ Hz$$

- The conductivity is in a classical Drude model ~~of electron transport~~:

$$\sigma = \omega_p^2 \tau_c = \frac{ne^2}{m}\tau_c$$

$$\simeq \omega_p \underbrace{(\omega_p \tau_c)}_{\sim 100} \sim 10^{18}\ \frac{1}{s}$$

(3) For quasi-statics in metals we need

$$\omega \ll c/L \leftarrow \text{quasi-static}$$

$$\left.\begin{array}{l} \omega \ll 1/\tau_c \\ \omega \ll \omega_p \end{array}\right\} \leftarrow \omega \text{ less than micro frequencies}$$

Saw that a new scale arose:

$\omega \longrightarrow$

$\omega < \omega_{ind}$	$\omega > \omega_{ind}$	quasi-static macroscopic E+M doesn't apply
induced ~~currents~~ are small magnetic fields are not screened on length L	induced currents screen magnetic fields on scale $\delta = \sqrt{\frac{2c}{\sigma\omega}}$, $\delta < L$	$\omega \sim \tau_c$, $\omega \sim \frac{L}{c}$, $\omega \sim \omega_p$ because frequencies are large.

$$\omega_{ind} \sim \frac{c^2}{L^2}\frac{1}{\sigma}$$

(4) Formally showed that B satifies a diffusion equation:

$$-\nabla^2 B = \frac{\sigma}{c^2}\partial_t B$$

These terms are equal when $\omega \sim \omega_{ind}$

$$\frac{B}{L^2} \sim \frac{\sigma}{c^2} \omega B$$

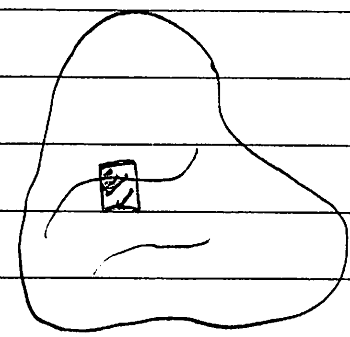
$$\boxed{\underbrace{\frac{c^2}{L^2 \sigma}}_{\omega_{ind}} \sim \omega}$$

Today

- Summarize Maxwell equations and talk about waves

Energy Conservation for Simple Matter

The energy density



- $u_{TOT} \equiv$ the total energy per volume

- $u_{EM} \equiv$ the electric + magnetic energy per vol

$$u_{EM} = \underbrace{\frac{1}{2}\vec{E}\cdot\vec{D}}_{\equiv u^{E}_{EM}} + \underbrace{\frac{1}{2}\vec{H}\cdot\vec{B}}_{\equiv u^{B}_{em}}$$

- u_{mech} = mechanical energy per vol.

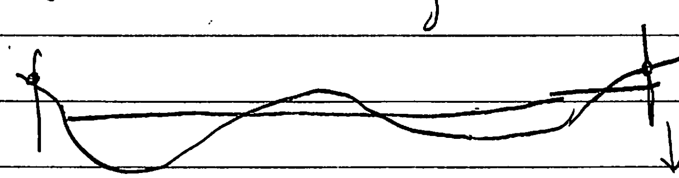
- The Total energy flux. $\vec{S}_{TOT}$. Expect:

$$\boxed{\partial_t u_{TOT} + \partial_i S^i_{TOT} = 0}$$

In this way an isolated system will conserve energy

- The Energy flux has mechanical pieces: S_{mech}

→ Take a stretched string for example



$$S_{string} = \underbrace{T_0 \frac{\partial y}{\partial x}}_{\text{Force}} \times \underbrace{\frac{\partial y}{\partial t}}_{\text{Velocity}}$$

→ S_{mech} records how energy is mechanically transported from one part to the other;

- The electromagnetic flux S_{em}:

$$\boxed{S_{em} = c\,(\vec{E} \times \vec{H})}$$

poynting vector

So expect:

$$\partial_t (u_{mech} + u_{em}) + \partial_i (S^i_{mech} + S^i_{em}) = 0$$

In integral form: $U = \int u \, dV$

$$\frac{d(U_{mech} + U_{em})}{dt} = -\int \vec{S}_{em} \cdot d\vec{a} - \cancel{\oint \vec{S}_{mech} \cdot d\vec{a}}$$

0 for an isolated mechanical system

Prf

$$\partial_t u_{mech} + \partial_i S^i_{mech} = \underbrace{\vec{j} \cdot \vec{E}}_{\text{work done per vol}}$$

Now

$$\vec{j} \cdot \vec{E} = c\left(\nabla \times H - \frac{1}{c}\partial_t D\right) \cdot E$$

$$= \underbrace{c\left(\nabla \times H \cdot \vec{E}\right)}_{c\nabla\cdot(H\times E) + H\, \underbrace{c\nabla\times E}} - \vec{E} \cdot \partial_t \vec{D}$$

$$= -\nabla \cdot S_{em} - H(\partial_t B) - E\partial_t D$$

$$\vec{j} \cdot \vec{E} = -\nabla \cdot S_{em} - \underbrace{\left(E\partial_t D + H\partial_t B\right)}_{\partial_t u_{em}}$$

So

$$\vec{j} \cdot \vec{E} = -\nabla \cdot S_{em} - \partial_t u_{em}$$

And

$$\boxed{\partial_t u_{mech} + \partial_i S^i = -\partial_t u_{em} - \partial_i S^i_{em}}$$

Momentum Conservation:

$$\partial_t g^j_{TOT} + \partial_i T^{ij}_{TOT} = 0$$

g^j_{TOT} = is the momentum per volume

T^{ij} = is the force in the j-th direction per area in the i-th direction.

If $\vec{g}$ obeys a conservation law, then total momentum of an isolated system is conserved:

$$\frac{dP^j_{TOT}}{dt} = \int dV\, \partial_t g^j_{TOT} = \int -\partial_i T^{ij}\, dV$$

$$= \int_{\partial V} T^{ij}\, \vec{n}_i\, dS = 0$$

for an "isolated" system

- $g^j_{TOT} = g^j_{mech} + g^j_{em}$
- $T^{ij}_{TOT} = T^{ij}_{mech} + T^{ij}_{em}$

$$\boxed{\vec{g}_{em} = \frac{\varepsilon\mu}{c^2} \vec{S}_{em}}$$

- $$T^{ij}_{em} = -\frac{1}{2}\left(E^i D^j + D^i E^j - \vec{D}\cdot\vec{E}\,\delta^{ij}\right) + -\frac{1}{2}\left(H^i B^j + H^j B^i - H\cdot B\,\delta^{ij}\right)$$

$$= \varepsilon\left(-E^i E^j + \frac{1}{2}E^2 \delta^{ij}\right) + \frac{1}{\mu}\left(-B^i B^j + \frac{1}{2}B^2 \delta^{ij}\right)$$

So the full relsult is

$$\partial_t g_{mec} + \partial_t g_{em} + \partial_i T^{ij}_{mech} + \partial_i T^{ij}_{em} = 0$$

For an mechically isolated system, $\vec{P} = \int \vec{g}\, dV$:

$$\frac{d}{dt}\left(P_{mech} + P_{em}\right) = -\int dS\, n_i T^{ij}_{em} - \underbrace{\cancel{\int dS\, n_i T^{ij}_{mec}}}_{\substack{\text{zero} \\ \text{for} \\ \text{mechanicall} \\ \text{isolated systems}}}$$

Recall that $-\partial_i T^{ij}_{mech}$ is the force per volume $\equiv \vec{f}_{mech}$

So this says

$$\frac{\partial}{\partial t} g^j_{mech} + \underbrace{\frac{\partial T^{ij}_{mech}}{\partial x^i}}_{-f^j_{mech}} = f^j_{em}$$

Prf

$$f_{em} = \rho E^j + \left(\frac{\vec{j}}{c} \times \vec{B}\right)^j$$

Now write, $\rho = \nabla \cdot E$ and $\frac{\vec{j}}{c} = \nabla \times H - \frac{1}{c}\partial_t \vec{D}$,

Find

$$f^j_{em} = \underbrace{(\nabla \cdot D) E^j}_{1} + \underbrace{\left[(\nabla \times H) \times \vec{B}\right]^j}_{2} - \underbrace{\frac{1}{c}(\partial_t \vec{D} \times \vec{B})^j}_{3}$$

The rest is labor and I will not go through

$$1 \Rightarrow -\partial_i T^{ij}_{E} \quad \text{with} \quad \boxed{\begin{aligned} T^{ij}_{E} &= -\varepsilon E^i E^j + \frac{1}{2} E^2 \delta^{ij} \\ T^{ij}_{B} &= -\frac{1}{\mu} B^i B^j + \frac{1}{2\mu} B^2 \delta^{ij} \end{aligned}}$$

$$2 \Rightarrow -\partial_i T^{ij}_{B} \quad \text{with}$$

$$3 \Rightarrow -\partial_t \vec{g} \quad \text{with} \quad \boxed{\vec{g}_{em} = \frac{\varepsilon\mu}{c^2} \vec{S}} = \frac{\varepsilon\mu}{c} \vec{E} \times \vec{H}$$

Angular Momentum Conservation

$$\partial_t g^k + \partial_\ell T^{\ell k} = 0 \quad \Leftarrow \text{momentum conservation}$$

And take $T^{\ell k} = T^{k\ell}$ symmetric

$$\underbrace{\partial_t ((r \times g)_i)}_{\text{ang momentum}} + \varepsilon_{ijk} r^j \partial_\ell T^{\ell k} = 0$$

Now $\varepsilon_{ijk} r^j \partial_\ell T^{\ell k} = \varepsilon_{ijk} \left[\partial_\ell (r^j T^{\ell k}) - \underbrace{\delta^j_\ell T^{\ell k}}_{=T^{jk}} \right]$ with $\delta^j_\ell = \frac{\partial r^j}{\partial r^\ell}$

So since $\varepsilon_{ijk} T^{jk} = 0$, we have

$$\boxed{\partial_t ((\vec{r} \times \vec{g})_i) + \partial_\ell (\varepsilon_{ijk} r^j T^{k\ell}) = 0}$$

So the angular momentum of the system is conserved, provided the stress tensor is symmetric

$$\boxed{\vec{L}_{field} = \int_V \vec{r} \times \vec{g}_{em}}$$

$$\frac{d}{dt}\left(\vec{L}_{mech} + \vec{L}_{field}\right)_i = \underbrace{-\int_{\partial V} dS\ \varepsilon_{ijk}\, r^j\, T^{kl}\, n_l}_{\text{net torque exerted on system}}$$