Waves at Higher Frequency - Dispersion

\[ \nabla \cdot E = \rho_{\text{mat}} \]

\[ \nabla \times B = \frac{j_{\text{mat}}}{c} + \frac{1}{c} \partial_t E \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = -\frac{1}{c} \partial_t B \]

Generally, have been assuming \( \omega \ll \frac{1}{\tau_{\text{micro}}} \)

\( k \ll \frac{1}{l_{\text{micro}}} \) or \( \lambda \gg l_{\text{micro}} \)

Certainly this is far from clear in the optical range

\[ t_{\omega} = \frac{\hbar c}{\omega} = \frac{\hbar c}{\frac{2\pi}{\lambda}} = \frac{197 \text{ eV} \cdot \text{nm}}{600 \text{ nm}} \]

for \( \lambda = 600 \text{ nm} \)

\[ t_{\omega} = 2.0 \text{ eV} \text{ of order atomic energies} \]
However, note

\[ \lambda \sim 600 \text{ nm} \sim 6000 \text{ Å} \]

That \( \lambda \gg \text{atomic sizes} \sim 0.5 \text{ Å} \)

So we can still expand the current in spatial gradients but need

Atoms in crystal visible light

to consider the atomic response times.

\[ \nabla \cdot \mathbf{E} = \rho(t) \]

\[ \nabla \times \mathbf{B} = \mathbf{j}(t)/c \]

\[ \nabla \cdot \mathbf{B} = 0 \]

\[ \nabla \times \mathbf{E} = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t} \]

What is \( \mathbf{j}_{\text{mat}} \)?
Linear Response for \( j_{\text{mat}}(t) \)

In general:

- \( j(t,x) \) depends on the fields.
  Work in a linear approximation.

Then the most general form involving no derivatives is

\[
\vec{j}(t) = \int dt' \ \sigma(t - t') \ \vec{E}(t') + \text{spatial}\end{equation}

\[
\text{response function}\end{equation}

Clearly, for a causal system, \( j(t) \) depends on \( E(t') \) for \( t' < t \). Thus, at minimum

\[
\sigma(t) = 0 \quad \text{for} \quad t < 0
\]

Then in frequency space, since the Fourier transform of a convolution is a product:

\[
\vec{v}(\omega) = \sigma(\omega) \ \vec{E}(\omega)
\]

\[
\text{frequency dependent conductivity}
\]
General Expectations for $\sigma(\omega)$:

1. For a conductor, $\mathbf{j} = \sigma \mathbf{E}$
   
   for $\omega \ll 1 / \tau_{\text{micro}}$. And thus
   
   $$\sigma(\omega) \approx \sigma_0$$
   at small frequencies.

2. For an insulator

   $\mathbf{j} = 2\epsilon \mathbf{P}$ at small frequencies

   $\mathbf{j}(\omega) = -i\omega \mathbf{P}$

   $\mathbf{j}(\omega) = -i\omega \chi_e \mathbf{E}$ at small frequencies

   Thus we sometimes define for insulators

   $\sigma(\omega) \equiv -i\omega \chi_e(\omega)$

   and $\sigma(\omega) \mathbf{E}(\omega) \equiv -i\omega \mathbf{P}(\omega)$
Maxwell Eqs & Dispersion

Can continue and add the first derivatives:

\[ j(\omega) = -i\omega \chi_e(\omega) E(\omega) + c \chi_m(\omega) \nabla \times B(\omega) \]

Then from \( \partial_t \rho + \nabla \cdot j = 0 \), \( \rho(\omega) = \frac{\nabla \cdot j}{-i\omega} \)

we have:

\[ \rho(\omega) = -\chi_e(\omega) \nabla \cdot E \]

Thus the only difference between this and before is that now \( \chi_e \) and \( \chi_m \) are functions of \( \omega \), always complex functions.

Find

\[ \varepsilon(\omega) \nabla \cdot E = 0 \]

\[ \nabla \times B = \frac{\varepsilon(\omega) \mu(\omega)}{c^2} (-i\omega E) \]

\[ \nabla \cdot B = 0 \]

\[ \nabla \times E = \frac{+i\omega B}{c} \]

Same as before but now \( \varepsilon(\omega) \) is a function of \( \omega \).
Maxwell Eqs. & Dispersion pg. 2

Look for plane wave solutions

\[ \mathbf{E}(x) = E_0 \, e^{i k \cdot x} \]

Then:

\[ \varepsilon(\omega) \mathbf{k} \cdot \mathbf{E}_0 = 0 \quad \Rightarrow \quad \mathbf{E}_0 \text{ is transverse unless } \varepsilon(\omega) = 0 \]

\[ \mathbf{k} \times \mathbf{B}_0 = \varepsilon \mu (-\omega \mathbf{E}_0) \]

\[ \mathbf{k} \cdot \mathbf{E}_0 = 0 \]

\[ \mathbf{k} \times \mathbf{E}_0 = \omega \mathbf{B}_0 \]

Transverse modes: \[ \mathbf{E}_0 \cdot \mathbf{k} = 0 \]

\[ \mathbf{k} \times (\mathbf{k} \times \mathbf{E}) = \frac{\omega}{c} \mathbf{k} \times \mathbf{B} \]

\[ \mathbf{k} \cdot (\mathbf{k} \times \mathbf{E}) = -k^2 \mathbf{E}_0 = -\frac{\omega^2}{c^2} \varepsilon(\omega) \mu(\omega) \mathbf{E}_0 \]

\[ k^2 \mathbf{E}_0 = \frac{\omega^2}{c^2} \varepsilon(\omega) \mu(\omega) \mathbf{E} \]

Determines

\[ k^2 = \omega^2 \, \frac{n^2(\omega)}{c^2} \]

Complex and frequency dep

\[ n^2(\omega) = \varepsilon(\omega) \mu(\omega) \]

Index of refraction
Propagation of Waves in Dispersive Medium

- The real part of $\varepsilon(\omega)$ can be interpreted as the index of refraction
- The imaginary part is absorption

To see this set $\mu = 1$, then the index of refraction is complex

$$n = [\text{Re } n] + i[\text{Im } n] = \sqrt{\varepsilon} \approx \frac{\sqrt{\text{Re } \varepsilon}}{2 \text{ Re } \varepsilon} (1 + i \frac{\text{Im } \varepsilon}{2 \text{ Re } \varepsilon})$$

Then looking at the Helmholtz equation:

$$[\nabla^2 + \left(\frac{\omega n}{c}\right)^2] E = 0$$

with a trial solution $E = A e^{i \kappa x}$, we see that:

$$-\kappa^2 + \left(\frac{\omega n}{c}\right)^2 = 0 \quad \kappa = \frac{\omega}{c} (\text{Re } n + i \text{ Im } n)$$

And then

$$E_T \sim A e^{i \frac{\omega}{c} [\text{Re } n] x} e^{-\frac{\omega}{c} [\text{Im } n] x}$$
Simple model for $\sigma(\omega)$ for a dielectric

Let's go back and revive the oscillator model.

Atoms electrons harmonically bounded to protons

$$m \frac{d^2x}{dt^2} + m \gamma \frac{dx}{dt} + mw_0^2 = eE \frac{e^{-i\omega t}}{\omega}$$

Find $x = x_0 e^{-i\omega t}$

$$[-mw^2 = i m \gamma \omega + mw_0^2] \times \omega = eE \omega$$

So

$$x_0 = \frac{(e/m)E \omega}{-\omega^2 + w_0^2 - i \omega \gamma}$$

And $j\omega = eN(-i\omega)x_0$  $j = eN V(t)$

$$j\omega = \frac{(Ne^2/\mu)}{-\omega^2 + w_0^2 - i \omega \gamma}$$
So, the Lorentz model for a dielectric is given by:

\[ \chi_e(\omega) = \frac{(Ne^2/m)}{-\omega^2 + \omega_0^2 - i\omega\gamma} \]

Find:

\[ \chi_e(\omega) = \text{Re} \chi_e + i \text{Im} \chi_e \]

\[ \chi_e = \frac{(Ne^2/m)}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \]

\[ \text{Re} \chi_e = \frac{(Ne^2/m)}{(\omega_0^2 - \omega^2)^2 + (\omega\gamma)^2} \]

So,

\[ \varepsilon = 1 + \chi_e \]

\[ \text{Re} \varepsilon(\omega) = 1 + \text{Re} \chi_e \]

\[ \text{Im} \varepsilon(\omega) = \text{Im} \chi_e \]
Wave Packets

- So far we have been considering individual plane waves. A general wave is a superposition of plane waves.

The wave packet should also be a solution to the equations of motion, which means for every \( k \) there is an \( \omega(k) \). We will assume \( \omega(k) \) real. In general \( \omega(k) = i \pi / 2(k) \).

Then:

\[
U(x,t) = \int_{-\infty}^{\infty} dk \ A(k) \ e^{i k x - \omega(k) t}
\]

The shape of the initial packet determines \( A(k) \).

\[
U(x,0) = \int_{-\infty}^{\infty} dk \ A(k) e^{i k x} \Rightarrow A(k) = \int_{-\infty}^{\infty} dx \ U(x,0) e^{-i k x}
\]
Wave packets p.g. 2

Then the uncertainty relation says that

$$\Delta k \Delta x \geq \frac{1}{2}$$

Where

$$\langle \Delta k \rangle^2 = \int \frac{dk}{2\pi} |A(k)|^2 (k-E)^2$$

$$\langle \Delta x \rangle^2 = \int dx |\psi(x,0)|^2 (x-x_0)^2$$

For a wave shown:

$\psi(x)$

* The width $\Delta k \ll k_0$
Wave Packets (pg. 3)

Then

\[ u(x, t) = \int_{-\infty}^{\infty} A(k) e^{i k x - i \omega(k) t + \frac{\omega(k) - \omega(k_0)}{2 \pi} k^2} e^{-i \phi} \]

And we expand

\[ \omega(k) \approx \omega(k_0) + \frac{d \omega}{d k}(k - k_0) t + \ldots \]

So

\[ u(x, t) = e^{i \phi_0 t} \int_{-\infty}^{\infty} A(k) e^{i k (x - \frac{d \omega(k_0)}{d k} t)} \]

Thus we see that apart from an irrelevant phase, the wave packet travels with a speed given by

\[ v_g = \frac{d \omega}{d k} \Big|_{k_0} \]
Wave packets

For

\[ w(k) = \frac{ck}{n(k)} \]

\[ \frac{dw}{dk} = c - \frac{ck}{n(k)} \frac{dn}{dw} \frac{dw}{dk} \]

So \[ \frac{dw}{dk} = \frac{C}{n(w)} \]

\[ \frac{1}{1 + \frac{w}{n} \frac{dn}{dw}} \]

\[ \frac{dw}{dk} = \frac{c}{n(w) + \frac{dn}{dw}} \]
Problem

- When analyzing the reflection of light off metal:

\[
\begin{align*}
H_1 & \quad \text{mm} \quad H_2 \\
\text{mm} & \\
\end{align*}
\]

We showed that:

\[
H_1 = H_1 e^{ikz-i\omega t} + H_2 e^{ikz+i\omega t}
\]

where

\[
\frac{H_2}{H_1} = \frac{1 - \sqrt{\frac{4\mu_0}{\sigma}}}{1 + \sqrt{\frac{4\mu_0}{\sigma}}}
\]

\[
= 1 - \sqrt{\frac{2\mu_0}{\sigma}} - i \sqrt{\frac{2\mu_0}{\sigma}}
\]

\[
\approx (1 - \frac{\sqrt{2\mu_0}}{\sigma}) e^{i\phi}
\]

- where \( \tan \phi \approx \sin \phi \approx \phi = \sqrt{\frac{2\mu_0}{\sigma}} \)
Now study a wave packet propagating into the metal.

- Show that the phase is irrelevant for the reflection coefficient.

- BUT show that the phase causes to a time delay between the naive (geometric optic) arrival time and actual arrival time of the reflected pulse. Compute the time delay.

- Interpret your result:

  \[ \text{distance} = L \]

  \[ \Delta t = \frac{2L}{c} + \text{bit} \]
Solution:

The incoming wave packet has

\[ H_I(x,t) = \int \frac{dk}{2\pi} e^{ikx - i\omega t} H_I(k) \]

The phases are chosen so that the pulse hits the metal at \( z = 0 \) and \( t = 0 \)

Then

\[ H_R(x,t) = \int \frac{dk}{2\pi} e^{-ikx - i\omega t + \phi(k)} H_I(k) R(k) e^{i\phi(k)} \]

where

\[ R(k) = 1 - \sqrt{\frac{2\mu \omega}{\sigma}} \quad \phi(k) = \sqrt{\frac{2\mu \omega}{\sigma}} \]