Last Time

- Talk about the retarded Green's function of SHO

\[ \frac{md^2 + m\gamma \frac{d}{dt} + mw^2}{dt^2} \] \[ G_R(t,t_0) = \delta(t - t_0) \]

- Uses describes the response to a force

\[ x(t) = \int G_R(t-t_0) F(t_0) \, dt_0 \iff x(\omega) = G_R(\omega) F(\omega) \]

- Since it is causal \( G_R(t) = 0 \) for \( t < 0 \):

\[ G_R(\omega) = \int_0^\infty e^{i\omega t} G_R(t) \]

- Discussed how to calculate the retarded Green's function

\[ G_R(t) = e^{\gamma t} \frac{1}{m} \sin \omega t \Theta(t) \quad \text{for } \eta \to 0 \]

\[ G_R(\omega) = \frac{1}{m} \frac{1}{\omega^2 + \omega_0^2 - i\omega \eta} \]
Why talk about this?

- The retarded grn fn of SHO is closely related to the wave eqn.

\[ \mathcal{G}_r(w) \]

2. Causal functions such as the retarded grn-fcn, the conductivity \( \sigma(w) \), \( \kappa_e(w) \), \( \varepsilon(w) = 1 + \kappa_e(w) \) are all analytic in the upper half plane. This allows us to relate the real and imaginary parts (Today? )
A comment about δ-funcs

Let's think about the identity

\[ \int \frac{e^{ikx}}{2\pi} \, dk = \delta(x) \leftarrow \text{what does it mean?} \]

First of all we always think about δ-funs
as a limit of a sequence of funcs

\[ \text{Should write } \delta_\varepsilon(x), \text{ where} \]

\[ \varepsilon \text{ is the width and think about } \varepsilon \to 0. \]

The uncertainty principle says the width
in momentum \( \xi \) is of order \( \Delta = \frac{1}{\varepsilon} \).

Thus we think about \( \Delta \) as a limit.

Example 1 - Cut off integral at \( \Lambda \)

\[ \Lambda = \frac{1}{\varepsilon} \]

\[ \Lambda = -\frac{1}{\varepsilon} \]

\[ \int_{\Lambda}^{1/\varepsilon} e^{ikx} \, dk = \frac{\sin x/\varepsilon}{\pi x/\varepsilon} = \delta_\varepsilon(x) \]

\[ \delta_\varepsilon(x) \quad \text{transform of } \delta_\varepsilon(x) \]

\[ \text{fourier} \]

\[ \sin x/\varepsilon/(x/\varepsilon) \]

\[ \frac{1}{x/\varepsilon} \]
Example 2

\[
\int_{-\infty}^{\infty} \frac{ekx - e^{ikx}}{(2\pi)^2} = \frac{1}{\pi x^2 + \varepsilon^2} \quad \varepsilon \to 0
\]

Any regulator is fine provided the width of the function is large compared to \( \varepsilon \)

\[
\int d^3x \ f(x) S_\varepsilon(x-x_0) \approx f(x_0)
\]

In Fourier space e.g. for \( S_\varepsilon(x) = \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \)

\[
f(k) e^{-\varepsilon |k|} \approx f(k)
\]

This is almost unity for a large range of \( k \) up to \( k \approx \Lambda \)

Moral

All integrals involving deltas should be thought of as a sequence of integrals
Green's Function of the Wave Eqn

\[ \Box u(t, x) = J(t, x) \tag{Source} \]

In \( E+M \) these will be currents

\[ \Box \equiv \frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \]

Example

\[ \Box \equiv -\frac{1}{c^2} \frac{\partial^2}{\partial t^2} + \nabla^2 \]

Currents acting as a source

\[ J \]

waves described by \( \nabla \)

The induced waves are

\[ u(t, x) = \int G_R(t-t_0, x-x_0) J(t_0, x_0) \]

also write \( G_R(t, x; t_0, x_0) \)

Then \( G_R(t, x; t_0, x_0) \) is the field at \( x \) due to a point source at \( t_0, x_0 \)

\[ \Box G_R(t, x; t_0, x_0) = \delta(t-t_0) \delta^3(x-x_0) \]

So
\[- \nabla u(t, x) = \int \left( - \nabla G_r(t \mid t_0, x) \right) J(t_0, x_0) \, dt_0 \, dx_0 \\delta(t-t_0) \delta(x-x_0) \]

\[- \nabla u(t, x) = J(t, x) \]

Solving for the Green's Func

\[
\begin{bmatrix}
\frac{1}{2} \frac{\partial^2}{\partial t^2} - \nabla^2 \\
\frac{1}{c^2} \frac{\partial^2}{\partial x^2}
\end{bmatrix}
G(t-t_0, x-x_0) = S(t-t_0) \delta^3(x-x_0)
\]

First choose \( t_0 = x_0 = 0 \):

\[
\begin{bmatrix}
\frac{1}{2} \frac{\partial^2}{\partial t^2} - \nabla^2 \\
\frac{1}{c^2} \frac{\partial^2}{\partial x^2}
\end{bmatrix}
G(t, x) = S(t) \delta^3(x)
\]

Now Fourier transform in space:
\[
\hat{G}(t, \mathbf{k}) = \int \hat{e}^{i \mathbf{k} \cdot \mathbf{x}} \, G(t, \mathbf{x})
\]

\[
\begin{bmatrix}
\frac{1}{2} \frac{\partial^2}{\partial t^2} + k^2 \\
\frac{1}{c^2} \frac{\partial^2}{\partial x^2}
\end{bmatrix}
\hat{G}(t, k) = \hat{S}(t)
\]

\[
\frac{1}{c^2} \left[ \frac{\partial^2}{\partial t^2} + \left( \frac{c}{k} \right)^2 \right] \hat{G}(t, k) = \hat{S}(t)
\]

Compare SHO
\[
m \left[ \frac{\partial^2}{\partial t^2} + \omega_0^2 \right] \hat{G}(t, k) = \hat{S}(t)
\]
Solving For Grn Fen pg 2

So the retarded green fen can be taken "lock, stock, and barrel" from SHO: with \( w_0 = ck \)

\[
G(t, k) = \frac{c^2 \Theta(t) \sin ckt}{ck}
\]

Now we only need to take the inverse FT:

\[
G(t, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{i \vec{k} \cdot \vec{r}} \frac{c^2 \Theta(t) \sin ckt}{ck}
\]

This integral is not convergent. But this should not surprise us. Add a convergence factor

\[
G(\varepsilon, \vec{r}) = \int \frac{d^3k}{(2\pi)^3} e^{-\varepsilon|\vec{k}|} e^{i \vec{k} \cdot \vec{r}} \frac{c^2 \Theta(t) \sin ckt}{ck}
\]

Satisfies

\[
-\nabla G(\varepsilon, \vec{r}) = \frac{S(t)}{\varepsilon} \delta^3(\vec{r}) \quad \text{as a limit as } \varepsilon \to 0
\]

To do the integral write \( R = |\vec{r}| \)

\[
G(\varepsilon, \vec{r}) = \int \frac{k^2dkd\phi}{(2\pi)^3} e^{-\varepsilon k} e^{i k R \cos \phi} \frac{c^2 \Theta(t) \sin ckt}{ck}
\]
Solving for Gen. Fcn. Pg. 3 - doing integrals

Do the angular integrals first

\[ \int d(\cos \theta) e^{ikR\cos \theta} = 2 \sin kR \]

So collecting all extraneous factors

\[ G_\varepsilon(t, \tau) = \frac{1}{2\pi^2 R} \int_0^\infty e^{-\varepsilon k} \sin kR \sin k\tau \]

Using

\[ \sin kR \sin k\tau = \frac{1}{2} \left[\cos (k(R-ct)) + \cos (k(R+ct))\right] \]

Then write

\[ \cos (k(R-ct)) = \frac{1}{2} \left[ e^{ik(R-ct)} + e^{-ik(R-ct)} \right] \]

and find

\[ \int_0^\infty dk e^{-\varepsilon k} \cos (k(R-ct)) = \frac{\varepsilon}{(R-ct)^2 + \varepsilon^2} \]

So

\[ G_\varepsilon = \frac{1}{2\pi R} \left[ \frac{1}{\pi} \frac{\varepsilon}{(R-ct)^2 + \varepsilon^2} - \frac{1}{\pi} \frac{\varepsilon}{(R+ct)^2 + \varepsilon^2} \right] \]

for \( \varepsilon \to 0 \) use

\[ s(x) = \lim_{\varepsilon \to 0} \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2} \]
Solving for green fcn pg. 4

Find

$$ G(t, R) = \frac{c}{4\pi R} \Theta(t) \left[ \frac{\delta(R - ct)}{2} + \delta(R + ct) \right] $$

\( t > 0 \quad R > 0 \)

\( \text{pull out } c \)

\( \frac{1}{c} \delta\left(\frac{R}{c} - t\right) \)

So

$$ G(t, R) = \frac{\Theta(t)}{4\pi R} \delta\left(\frac{R}{c} - t\right) $$

Picture:

More generally:

$$ G(t - t_0, \vec{r} - \vec{r}_0) = \frac{\Theta(t - t_0)}{4\pi |\vec{r} - \vec{r}_0|} S\left(\frac{|\vec{r} - \vec{r}_0| - (t - t_0)}{c}\right) $$

\( \text{we will use this next time} \)