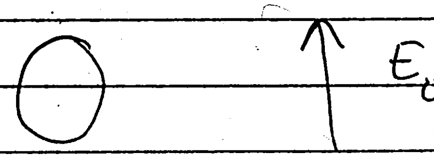
Solving the Laplace Eq by Separation

Problem

- Consider a conducting sphere of radius a in an external field



$\varphi = -E_0 z$
far from sphere

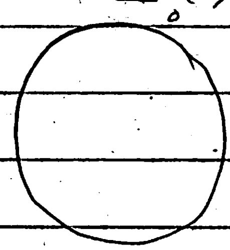
Find φ everywhere.

We want to find φ such that $\varphi = 0$ on bndry. First write:

$$\varphi = -E_0 r\cos\theta + \Phi$$

where $-\nabla^2 \Phi = 0$ and $\Phi = E_0 a\cos\theta$ at $r=a$. Then solve a prototypical problem

$\Phi_0(\theta, \phi) = E_0 a \cos\theta$, on bndry



What is $\Phi = ?$ outside.

Notice that if $\Phi = R(r)\, Y(\theta, \varphi)$

coordinate $\perp$ to surface

coords $\parallel$ to surface

Then we compute $\dfrac{-r^2}{\Phi} \nabla^2 \Phi$

$$\underbrace{-\frac{1}{R}\frac{\partial}{\partial r} r^2 \frac{\partial R}{\partial r}}_{\text{if } r \text{ is fixed}} + \underbrace{\frac{-1}{Y}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]Y}_{\text{Then this is constant}} = 0$$

Thus we are led to consider the eigenvalue eqn

$$\underbrace{\left[-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta}+\frac{-1}{\sin^2\theta}\frac{\partial^2}{\partial^2\phi^2}\right]}_{=L^2} Y_n = \lambda_n Y_n$$

This is the angular momentum squared operator of quantum mechanics. Since it is hermitian the eigenfcns are complete and orthogonal. And they are labelled by l,m and are known as spherical harmonics

$$L^2 Y_{lm}(\theta,\phi) = l(l+1) Y_{lm} \qquad \text{(eigen funcs)}$$

$$\int d\Omega\, Y_{lm}(\theta,\phi) Y_{l'm'}(\theta,\phi) = \delta_{ll'}\delta_{mm'} \qquad \text{(orthogonal)}$$

Thus we can expand the solution at each r

$$\Phi(r,\theta,\phi) = \sum_{lm} R_{lm}(r)\, Y_{lm}(\theta,\phi) \qquad \text{(complete)}$$

And adjust $R_{lm}(R)$ so the boundary cond are maintained

We have from previous page: $-\nabla^2\varphi = 0$

$$\left[-\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} + \frac{l(l+1)}{r^2}\right] R_{lm}(r) = 0$$

second order differential equation

Then solving

$$R_{\ell m} = A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}$$

We have

$$\boxed{\Phi = \sum_{\ell m} \left(A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}(\theta, \phi)}$$

these are adjusted to satisfy BC

For the current problem:

① $\Phi \xrightarrow[r \to \infty]{} 0$ implying $A_{\ell m} = 0$

② $\Phi \xrightarrow[r \to a]{} \Phi_0 = E_0 a \cos\theta$

On Boundary $r = a$: $\Phi = \Phi_0(\theta, \phi)$

$$\star \quad \sum_{\ell m} B_{\ell m} \frac{1}{a^{\ell+1}} Y_{\ell m}(\theta, \varphi) = E_0 a \cos\theta$$

Using orthoganility

$$\int d\Omega \, Y^*_{\ell m} Y_{\ell' m'} = \delta_{\ell \ell'} \delta_{m m'}$$

We multiply Eq $\star$ by $Y^*_{\ell m}$ and integrate

$$\frac{B_{\ell m}}{d^{\ell+1}} = \int d\Omega \; Y^*_{\ell m}(\theta,\phi) \; E_0 a \cos\theta$$

So since $E_0 a \cos\theta \propto Y_{10}$ we find that $B_{10} \neq 0$ and all others are zero. The remaining algebra shows

$$\varphi = -E_0 r\cos\theta + \Phi$$

$$\varphi = -E_0 r\cos\theta + \frac{E_0 a^3 \cos\theta}{r^2}$$

General Rule

(1) Identify coords perp (i.e. r) and parallel (θ,φ) to surface where boundary conditions are specified

(2) Solve eigenvalue eqn. for parallel directions these will be complete and or thogonal

(3) Expand the solution in these eigenfcns and solve for $\perp$ direction

$\perp$ direction

eigenfcns

$$\Phi = \sum_{\ell m} \left(A_{\ell m} r^\ell + \frac{B_{\ell m}}{r^{\ell+1}} \right) Y_{\ell m}$$

(4) ~~Adjust coefficients so that the boundary conditions are satified.~~

Comments on the Eigen-value eqn

We started by finding the eigenvectors Y_{lm} of the L^2 operator

$$\left[\frac{-1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{-1}{\sin^2\theta} \frac{\partial^2}{\partial\phi^2} \right] Y_{lm} = l(l+1) Y_{lm}$$

Now to solve; separate again $Y = \Theta(\theta)\,\Phi(\phi)$

$$-\frac{\partial^2 \Phi_m}{\partial^2\phi^2} = m^2 \Phi_m$$

$$\left[\frac{-1}{\sin\theta} \frac{\partial}{\partial\theta} \sin\theta \frac{\partial}{\partial\theta} + \frac{m^2}{\sin^2\theta} \right] \Theta_{lm} = l(l+1)\, \Theta_{lm}$$

or with $x = \cos\theta$

$$\left[-\frac{\partial}{\partial x} (1-x^2) \frac{\partial}{\partial x} + \frac{m^2}{1-x^2} \right] \Theta_{lm} = l(l+1)\, \Theta_{lm}$$

These equations are <u>all</u> of the <u>Sturm Louiville</u> type

$$\left[-\frac{d}{dx}\, p(x) \frac{d}{dx} + q(x) \right] \psi_n(x) = \lambda_n\, r(x)\, \psi_n(x)$$

with $p(x) > 0$ and $r(x) > 0$

Can Show for Sturm Louiville

- The eigen-fcns are complete, in the space of fcns satisfying the BC (Morse & Feshb. vol. 1 pg 736-738)
- The eigen fcns are orthogonal (homework)

$$\int \psi_n(x)\, \psi_m(x)\, r(x)\, dx = 0 \quad \text{if} \quad m \neq n$$

$r(x)$ acts as a weight

This is how you show (for example)

$$\int_0^\infty \rho\, d\rho\; J_m(k\rho)\, J_m(k'\rho) = 0 \quad \text{for} \quad k \neq k'$$

- The Sturm-Louiville Operator is self adjoint: (in space of fcns satisfying boundary conditions)

$$\mathcal{L} = \left[\frac{-d}{dx} p(x) \frac{d}{dx} + q(x) \right]$$

i.e. $$\int dx\, f(x) \left[\mathcal{L} g(x) \right] = \int dx \left[\mathcal{L} f(x) \right] g(x)$$

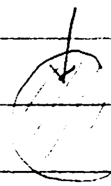
- The wronskian ~~of two solutions~~ $\times\, p(x)$ is constant (gauss La

$$p(x) \left[f(x)\, g'(x) - g(x)\, f'(x) \right] = \text{const}$$

is a statement of constant flux or Gauss Law

Example Showing why $p(x)\,\text{Wronsk}(x) = \text{const}$

Charge spherically symmetric



Free Space

in Free Space $\varphi = C + \dfrac{Q}{4\pi r} \Rightarrow E = \dfrac{Q}{4\pi r^2}$

- $\int \vec{E} \cdot d\vec{s} = Q$ gauss law

$$r^2 E_r = \text{const}$$

- Laplace eq:

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r}\right]\varphi = 0$$

→ Two solutions $f(r) = A$ $g(r) = \dfrac{B}{r}$

→ $p(r) = r^2$

$$p(r)\left[f(r)\,g'(r) - f'(r)\,g(r)\right] = -r^2 \underbrace{\left(\frac{B}{r^2}\right)}_{E_r} = \text{const}$$

Appendix C of Zangwill (and Wikipedia and DLMF) provide a useful summary of the special functions involved.

1 Cartesian coordinates: sec 7.5

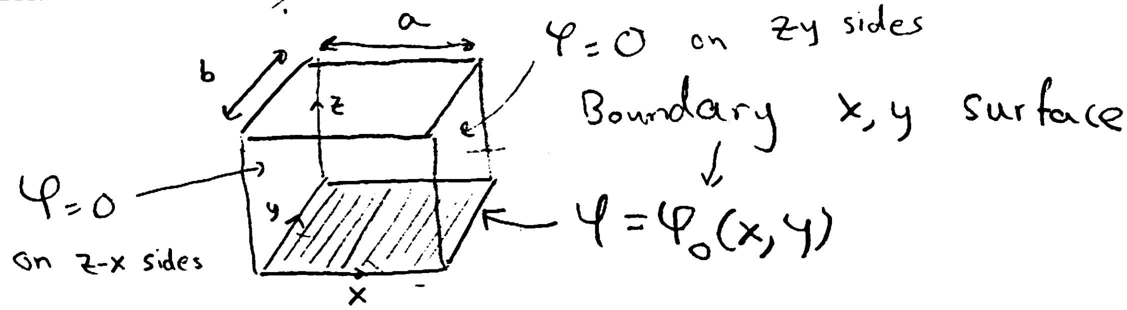


Figure: Boundary conditions for cartesian coordinates: sec 7.5

(a) Laplacian

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\varphi = 0 \tag{1}$$

(b) Eigen fucntions along boundary vanishing at $x = 0$ and $x = a$ and $y = 0$ and $y = b$

$$\psi_{nm}(x, y) = \sin\left(\frac{n\pi x}{a}\right)\sin\left(\frac{m\pi y}{b}\right) \qquad n = 1\ldots\infty \qquad m = 1\ldots\infty$$

(c) Orthogonality

$$\int_0^a dx \int_0^b dy\, \psi_{nm}\, \psi_{n'm'} = \left(\frac{a}{2}\right)\left(\frac{b}{2}\right)\delta_{nn'}\delta_{mm'}$$

(d) Solution

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\left[A_{nm}e^{-\gamma_{nm}z} + B_{nm}e^{+\gamma_{nm}z}\right]\psi_{nm}(x, y) \tag{2}$$

where $\gamma_{nm} = \sqrt{(n\pi/a)^2 + (m\pi/b)^2}$

2 Spherical coordinates: 7.6 and 7.7

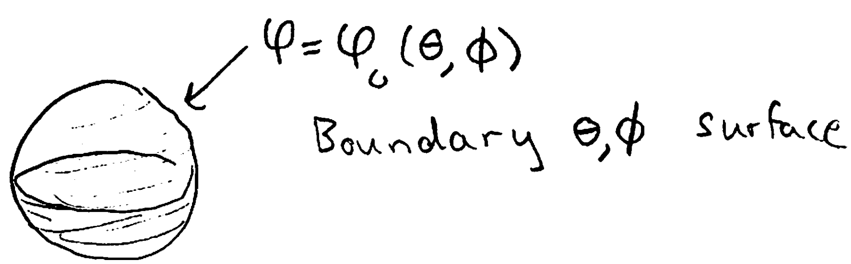


Figure: Boundary conditions in spherical coordinates: 7.6 and 7.7

(a) Laplacian

$$\left[\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right]\varphi = 0 \tag{3}$$

(b) Eigen fucntions along boundary θ, ϕ, regular at $\theta = 0$ and π, 2π periodic in ϕ

$$\psi_{\ell m}(\theta, \phi) = Y_{\ell m}(\theta, \phi) \qquad \ell = 0\ldots\infty \quad m = -\ell\ldots\ell$$

(c) Orthogonality:

$$\int d\Omega \, Y^*_{\ell m}(\theta,\phi)\, Y_{\ell' m'}(\theta,\phi) = \delta_{\ell\ell'}\delta_{mm'}$$

(d) Solution

$$\varphi = \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}\left[A_{\ell m} r^{\ell} + \frac{B_{\ell m}}{r^{\ell+1}}\right] Y_{\ell m} \tag{4}$$

(e) When there is no azimuthal dependence things simplify to

$$\varphi = \sum_{\ell=0}^{\infty}\left[A_{\ell} r^{\ell} + \frac{B_{\ell}}{r^{\ell+1}}\right] P_{\ell}(\cos\theta) \tag{5}$$

where $P_\ell(\cos\theta)$ is the legendre polynomial, which up to a normalization if $Y_{\ell 0}(\theta,\phi)$, satisfying the orthogonality

$$\int_{-1}^{1} d(\cos\theta) P_\ell(\cos\theta) P_{\ell'}(\cos\theta) = \frac{2}{2\ell+1}\delta_{\ell\ell'}$$

3 Cylindrical Boundary: z,ϕ are the boundary. Sec 7.8

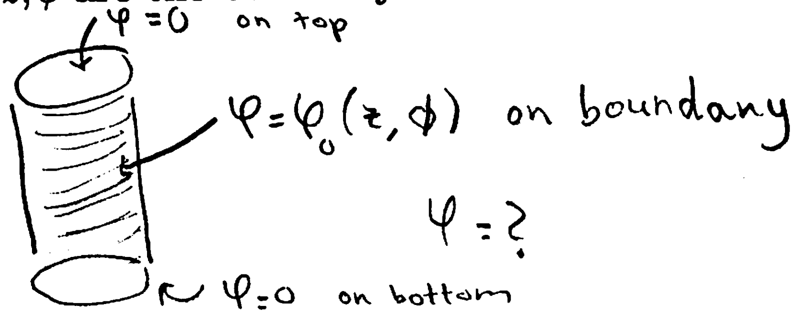


Figure: Boundary conditions in cylindrical coordinates with a z,ϕ boundary

(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\varphi = 0 \tag{6}$$

(b) Eigenfunctions along boundary z,ϕ vanishing at $z=0$ and $z=L$ and 2π periodic in ϕ

$$\psi_{nm}(z,\phi) = \sin(k_n z)\, e^{im\phi} \qquad k_n \equiv \frac{n\pi}{L} \qquad n = 1\ldots\infty \;\; m = -\infty\ldots\infty$$

(c) Orthogonality:

$$\int_0^L dz \int_0^{2\pi} \psi_{nm}(z,\phi)\,\psi_{nm}(z,\phi) = \frac{L}{2}(2\pi)\delta_{nn'}\delta_{mm'}$$

(d) Solution:

$$\varphi = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{nm} I_m(k_n\rho) + B_{nm} K_m(k_n\rho)\right]\psi_{nm}(z,\phi) \tag{7}$$

Here $I_\nu(x)$ and $K_\nu(x)$ is the modified bessel function of the first and second kinds. Note that $K_{-m}(x) = K_m(x)$ and $I_{-m}(x)$

4 2D cylindrical coordinates: sec 7.9

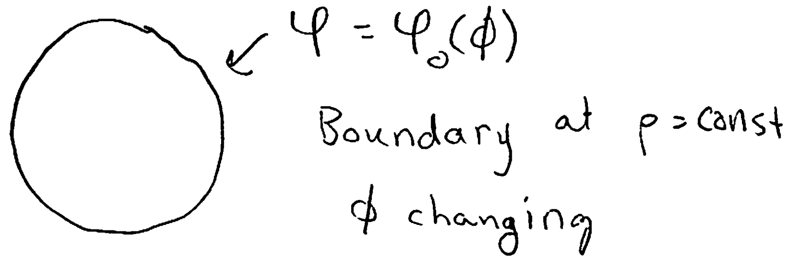


Figure: 2D Cylindrical coordinates: sec 7.9

(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2}\right]\varphi = 0 \tag{8}$$

(b) Eigenfunctions along boundary ϕ: 2π periodic in ϕ

$$\psi_m(\phi) = e^{im\phi} \qquad m = -\infty\ldots\infty$$

(c) Orthogonality

$$\int_0^{2\pi} \psi_m^*(\phi)\psi_{m'}(\phi) = 2\pi\delta_{mm'} \tag{9}$$

(d) Solution

$$\varphi = A_0 + B_0 \ln\rho + \sum_{m=-\infty}^{\infty}\left(A_m\rho^{|m|} + \frac{B_m}{\rho^{-|m|}}\right)\psi_m$$

5 Cylindrical Boundary: ρ, ϕ are the boundary – Sec 7.8

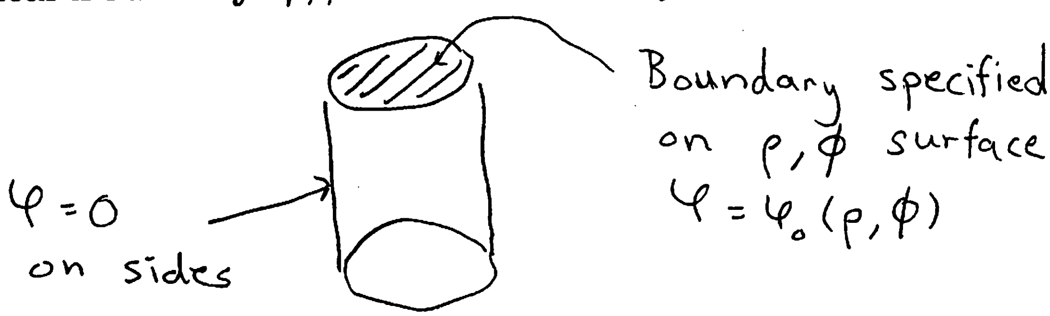


Figure: Boundary conditions in cylindrical coordinates with a ρ, ϕ boundary, sec 7.8

(a) Laplacian:

$$\left[\frac{1}{\rho}\frac{\partial}{\partial\rho}\rho\frac{\partial}{\partial\rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} + \frac{\partial^2}{\partial z^2}\right]\varphi = 0 \tag{10}$$

(b) Eigenfunctions along boundary ρ, ϕ vanishing at $\rho = R$ and regular at $\rho = 0$, 2π periodic in ϕ:

$$\psi_{mn}(\rho,\phi) = J_m(k_{mn}\rho)\, e^{im\phi} \qquad n = 1\ldots\infty \;\; m = -\infty\ldots\infty$$

Here:

$$k_{mn} = \frac{x_{mn}}{R} \tag{11}$$

where x_{mn} is the n-th zero of the m-th Bessel function, *e.g.* the zeros of $J_0(x)$ are

$$(x_{01}, x_{02}, x_{03}) = 2.40483,\ 5.52008,\ 8.65373 \tag{12}$$

These are given by x_{mn} = BesselZeroJ[m, n] in Mathematica. Note also that $J_{-m}(x) = J_m(x)$

(c) Orthogonality:

$$\int_0^R \rho d\rho \int_0^{2\pi} \psi_{mn}(\rho,\phi)\,\psi_{mn}(\rho,\phi) = \left(\frac{R^2}{2}\left[J_{m+1}(k_{mn}R)\right]^2\right)(2\pi)\,\delta_{nn'}\delta_{mm'}$$

(d) Solution:

$$\varphi = \sum_{n=1}^{\infty}\sum_{m=-\infty}^{\infty}\left[A_{mn}e^{-k_{mn}z} + B_{nm}e^{k_{mn}z}\right]\psi_{mn}(\rho,\phi) \tag{13}$$

6 Continuum Forms and Fourier and Hankel Transforms

In each case we are expanding a function in a complete set of eigenfunctions

$$\langle x|F\rangle = \sum_n \langle x|n\rangle\,\langle n|F\rangle \tag{14}$$

(a) For the cartesian case when a and b go to infinity. The sum becomes an integral and the sum over n and m becomes a 2D fourier transform

$$\varphi = \int \frac{d^2k_\perp}{(2\pi)^2} e^{ik_\perp\cdot x_\perp}\left[A(k_\perp)e^{-k_\perp z} + B(k_\perp)e^{k_\perp z}\right].$$

We are using the fact that any function in the x, y plane (in particular the boundary condition $\varphi_o(x,y)$) can be expressed as a fourier transform pairs

$$F(x,y) \equiv \int \frac{d^2k_\perp}{(2\pi)^2}\left[e^{ik_\perp\cdot x_\perp}\right] F(k_x,k_y), \tag{15}$$

$$F(k_x,k_y) \equiv \int d^2x_\perp \left[e^{-ik_\perp\cdot x_\perp}\right] F(x,y). \tag{16}$$

(b) For the cylindrical case when L goes to ∞, the sum over n becomes an integral yielding

$$\varphi = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\left[e^{i\kappa z}e^{im\phi}\right]\left[A(\kappa)I_m(|\kappa|\rho) + B(k)K_m(|\kappa|\rho)\right]$$

We are using the fact that any regular function of z and ϕ (in particular the boundary condition $\varphi_o(z,\phi)$) can be written in terms of its fourier components

$$F(z,\phi) = \sum_{m=-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{d\kappa}{2\pi}\left[e^{i\kappa z}e^{im\phi}\right] F_m(\kappa) \tag{17}$$

$$F_m(\kappa) = \int_0^{2\pi}\frac{d\phi}{2\pi}\int_{-\infty}^{\infty}dz\,\left[e^{-i\kappa z}e^{-im\phi}\right] F(z,\phi) \tag{18}$$

(c) Finally for the second cylindrical case when the radius goes to infinity

$$\varphi = \sum_{m=-\infty}^{\infty}\int_0^{\infty} kdk\,\left[J_m(k\rho)e^{im\phi}\right]\left[A(k)e^{-kz} + B(k)e^{kz}\right] \tag{19}$$

We are using the fact that any regular cylindrical function of ρ and ϕ (in particular the boundary condition $\varphi_o(\rho,\phi)$) can be written as *Hankel* transform

$$F(\rho,\phi) = \sum_{m=-\infty}^{\infty}\int_0^{\infty} kdk\,\left[J_m(k\rho)e^{im\phi}\right] F_m(k) \tag{20}$$

$$F_m(k) = \int_0^{2\pi}\frac{d\phi}{2\pi}\int_0^{\infty}\rho d\rho\,\left[J_m(k\rho)e^{-im\phi}\right] F(\rho,\phi) \tag{21}$$