Last Times

(1) $\nabla \cdot E = \rho(\omega)$

$\nabla \times B = \frac{\vec{j}(\omega)}{c} + \frac{-i\omega}{c}\vec{E}$

$\nabla \cdot B = 0$

$\nabla \times E = \frac{i\omega}{c}\vec{B}$

In order to specify the currents we postulated a constituent relation

$$j(\omega) = \sigma(\omega) E(\omega) \equiv -i\omega\chi_e(\omega) E(\omega)$$

Then we showed that the EOM are the same with $\varepsilon(\omega) = 1 + \chi_e(\omega)$. For wavelike solutions, $E(t,x) = \vec{E}_T e^{ikx - i\omega t}$, the Helmholtz equation, $\left[\nabla^2 + \frac{\omega^2 \varepsilon(\omega)}{c^2}\right]\vec{E} = 0$, becomes:

$$\left[-k^2 + \frac{\omega^2 \varepsilon(\omega)}{c^2}\right] = 0$$

$\leftarrow$ This defines the dispersion curve $\omega(k)$

So a general solution will be of the a superposition of waves of the form

$$E_T \, e^{-i\omega(k)t} e^{ik\cdot x}$$

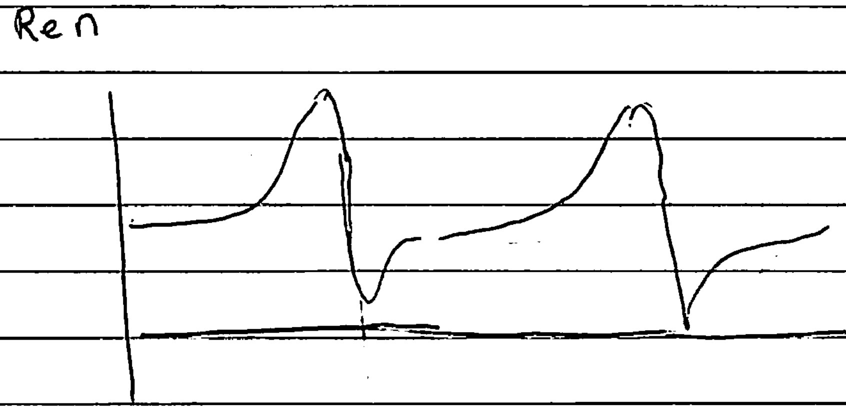
Where $\omega(k)$ has real and imaginary parts

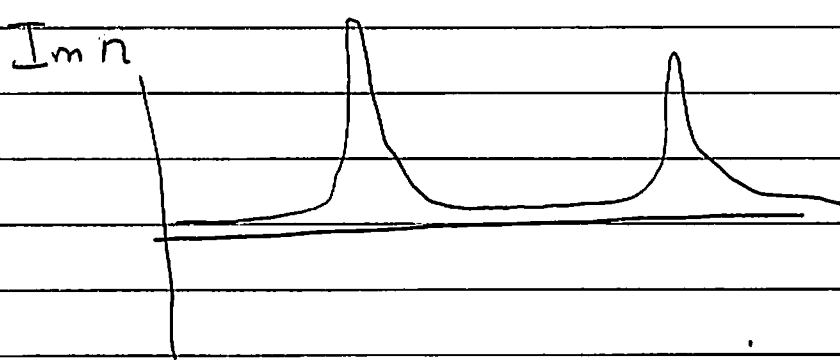
$$E_T \, e^{-i\,\mathrm{Re}\,\omega(k)t} \, e^{-[-\mathrm{Im}\,\omega(k)]t} \, e^{ikx}$$ (Im $\omega(k)$ will be negative)

For small imaginary part for $\varepsilon \equiv \varepsilon' + i\varepsilon''$:

$$\omega(k) \simeq \frac{ck}{\sqrt{\varepsilon'}}\left(1 - \frac{i\varepsilon''}{2\varepsilon'}\right) \qquad n \equiv \sqrt{\varepsilon} = n' + in''$$

So the imaginary part of ε records the absorption of the wave:





We had a simple model which described most of these features

- Most of the essential features are a consequence of causality. Since $\chi_e(t)$ is an causal (vanishes for $t<0$)

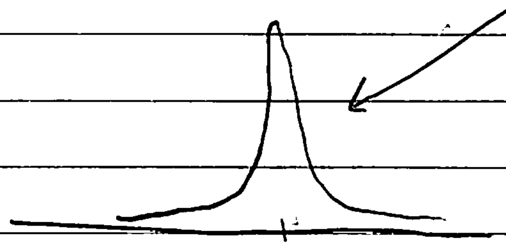
$$\chi_e(\omega) = \int_0^\infty e^{i\omega t} \chi_e(t)\, dt$$

$\chi_e(\omega)$ is analytic in the UHP. As such the Kramers-Kronig relations apply. The Kramers-Kronig relations apply to any causal function, e.g. $\sigma(\omega)$, $G_R(\omega)$, ... (provided the function decays sufficiently rapidly as $\omega \to \infty$)

Preliminaries for Kramers-Kronig

We talked about δ-fcns as a sequence of integrals

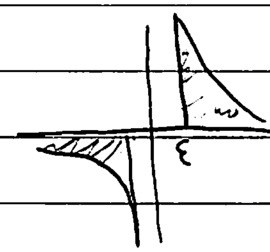
$$\int dx\, f(x)\, \delta(x) \equiv \lim_{\varepsilon \to 0} \int dx\, f(x) \frac{\varepsilon}{\pi}\frac{1}{x^2+\varepsilon^2} = f(0)$$



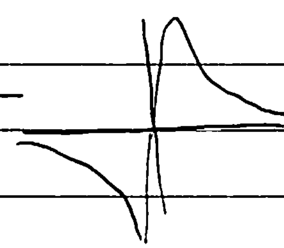
$x=0$

Similarly Principal value integrals are a limit of a seqence of fcns:

$$P \int dx \frac{f(x)}{x-x_0} = \underbrace{\int_{-\infty}^{x_0-\varepsilon} \frac{f(x)}{x-x_0} + \int_{x_0+\varepsilon}^{\infty} \frac{f(x)}{x-x_0}}_{\text{one definition}}$$



$$= \underbrace{\lim_{\varepsilon \to 0} \int dx\, f(x) \frac{(x-x_0)}{(x-x_0)^2+\varepsilon^2}}_{\text{another}}$$



So we get the identity:

$$\frac{1}{x+i\varepsilon} = \frac{x}{x^2+\varepsilon^2} - \frac{i\varepsilon}{(x^2+\varepsilon^2)} = P\frac{1}{x} - i\pi\,\delta(x)$$

Kramers-Kronig Relations

- What are they?
- For any function analytic in the upper half plane, i.e. the fourier transform of a causal function such as $\sigma(\omega)$, and $G_R(\omega)$, $\chi_e(\omega)$. This is a relation between the real and imaginary parts

$$\mathrm{Re}\,\chi_e(\omega_0) = \frac{2}{\pi} P \int_0^\infty \frac{\omega \,\mathrm{Im}\,\chi_e(\omega)}{\omega^2 - \omega_0^2}\, d\omega$$

$$\mathrm{Im}\,\chi_e(\omega_0) = -\frac{2\omega_0}{\pi} P \int_0^\infty \frac{\mathrm{Re}\,\chi(\omega)}{\omega^2 - \omega_0^2}\, d\omega$$

Thus, if you know the real part of the dielectric constant, then you know the imaginary (absorptive) part of the dielectric constant and vice-versa.

Also written, since $\chi_e(-\omega) = \chi_e^*(\omega) \Rightarrow$ Real part symmetric, Im part anti-symm.

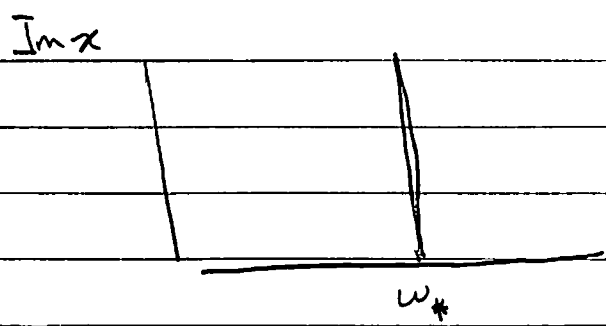
$$\mathrm{Re}\,\chi_e(\omega) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Im}\,\chi_e(\omega_0)}{\omega - \omega_0}$$

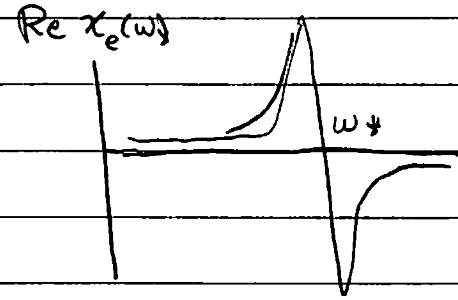
$$\mathrm{Im}\,\chi_e(\omega) = \frac{P}{\pi} \int_{-\infty}^{\infty} \frac{\mathrm{Re}\,\chi(\omega_0)}{\omega - \omega_0}$$

Uses of Kramers-Kronig

Thus, suppose $\operatorname{Im} \chi_e$ has a strong absorption peak at $\omega = \omega_*$:

$$\operatorname{Im} \chi(\omega) = C\,\delta(\omega - \omega_*)$$





Then the real part near for ω near ω_* is:

$$\operatorname{Re} \chi_e(\omega_0) \sim \frac{2}{\pi} P \int \frac{\omega\, C\, \delta(\omega - \omega_*)}{\omega^2 - \omega_0^2}\, d\omega$$

$$\sim \frac{2C}{\pi} \frac{\omega_*}{\omega_*^2 - \omega_0^2} \xrightarrow[\omega_0 \to \omega_*]{} \frac{-C}{\pi} \frac{1}{\omega_0 - \omega_*}$$

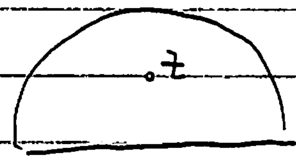
Thus we can begin to see why the SHO model for the dielectric function gave reasonable qualitative agreement; Most of the coarse features of the dielectric function are dictated by causality (i.e the Kramers-Kronig relations)

Krames - Kronig Relations Prf

- Since G is causal (and thus analytic) there exsist relations between the real and imaginary parts

Using Cauchy Thrm

$$G(z) = \oint \frac{G(\omega)}{\omega - z} \frac{d\omega}{2\pi i}$$



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Sending $z = \omega_0 + i\varepsilon$

$$G_R(\omega_0) = \int \frac{1}{2\pi i} d\omega \frac{G}{\omega - \omega_0 - i\varepsilon}$$

Using $\dfrac{1}{\omega - \omega_0 - i\varepsilon} = P \dfrac{1}{\omega - \omega_0} + i\pi\delta(\omega - \omega_0)$

Finding

$$G(\omega_0) = \left[\frac{P}{2\pi i} \int_{-\infty}^{\infty} d\omega \frac{G(\omega)}{\omega - \omega_0} \right] + \frac{1}{2} G(\omega_0)$$

Find

$$\operatorname{Re} G(\omega_0) = \frac{P}{\pi} \int_{-\infty}^{\infty} d\omega \, \frac{\operatorname{Im} G_R(\omega)}{\omega - \omega_0}$$

$$\operatorname{Im} G(\omega_0) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega \, \frac{\operatorname{Re} G_R(\omega)}{\omega - \omega_0}$$

Using the relation

$$G_R^*(\omega) = \int_0^{\infty} e^{-i\omega t} G_R(t) = G_R(\omega)$$

One finds

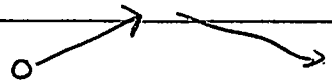
$$\operatorname{Re} \chi_e(\omega_0) = \frac{2P}{\pi} \int_0^{\infty} \frac{\omega \operatorname{Im} \chi_e(\omega)}{\omega^2 - \omega_0^2} \, d\omega$$

$$\operatorname{Im} \chi_e(\omega_0) = -\frac{2\omega_0 P}{\pi} \int_0^{\infty} \frac{\operatorname{Re} \chi_e(\omega)}{\omega^2 - \omega_0^2} \, d\omega$$

☆☆ Thus if you know the absorptive part of the dielectric constant, can determine the real part of dielectric constant and vice versa

Optics of Metals

We described a metal with a Classical transport (Drude Model)



- Electrons are free but experience a drag force

$$m\frac{dV}{dt} + \frac{m}{\tau_c}V = eE(t)$$

Found the D.C. conductivity

$$\sigma_0 = \omega_p^2 \tau_c \qquad \omega_p \sim 10^{15}\ 1/s \equiv \frac{n e^2}{m}$$

n = number of electrons per vol.

$$\simeq 10^{18}\ \frac{1}{s} \qquad \tau_c \sim 10^{13}\ s$$

τ_c = time between collisions with impurities

Excercise 1 Show that the constient relation in this model is

$$j(\omega) = \sigma(\omega)\, E(\omega)$$

is $$\sigma(\omega) = \frac{\sigma_0}{1 - i\omega\tau_c}$$

Excercise 2

- Determine the low frequency limits of $\varepsilon(\omega)$ and $\sigma(\omega)$
- Sketch the real and imaginary parts of $\varepsilon(\omega)$ for a typical metal
- Describe what the high frequency behavior means for the propagation of light
- Such a Drude model gives a not too unrealistic picture of $\varepsilon(\omega)$

Excercise 3

- Show that charge relaxation is govened by the equation

$$\partial_t \rho + \int \sigma(t-t')\,\rho(t')\,dt' = 0$$

Excercise 4

- Show that $\sigma(t-t')$ is determined up to a const by the green fcn of the differential equation

$$\left[m\frac{d}{dt} + \frac{m}{\tau_c} \right] G_R(t,t_0) = \delta(t-t_0)$$

Excercise 5

- Determine $G_R(t, t_0)$ by direct integration in time, and infer $\sigma(t-t')$. What time scale controls the relaxation of charge, σ, ω_p, τ_c? Estimate numerically

Excercise 6

- By fourier transforming $\sigma(\omega)$ determine $\sigma(\tau)$ where $\tau \equiv t-t'$

- Use complex variables and explain the differences between $\tau>0$ and $\tau<0$

Solutions

Exercise1 . Fourier transforming the equation

$$-im\omega V(\omega) + \frac{m}{\tau_c} V(\omega) = eE(\omega)$$

$$j(\omega) = N e V(\omega) = \frac{Ne^2/m}{-i\omega + \frac{1}{\tau_c}} E(\omega)$$

So $$j(\omega) = \frac{(Ne^2/m)\,\tau_c}{1 - i\omega\tau_c} E(\omega)$$

$$= \frac{\sigma_0}{1 - i\omega\tau_c}$$

Exercise2 For small frequency :

$$\mathcal{E}(\omega) = 1 + \chi_e(\omega) \simeq 1 + \frac{i\sigma_0}{\omega}$$

For general frequencies :

$$\mathcal{E}(\omega) = 1 + \frac{i\sigma_0/\omega}{1 - i\omega\tau_c}$$

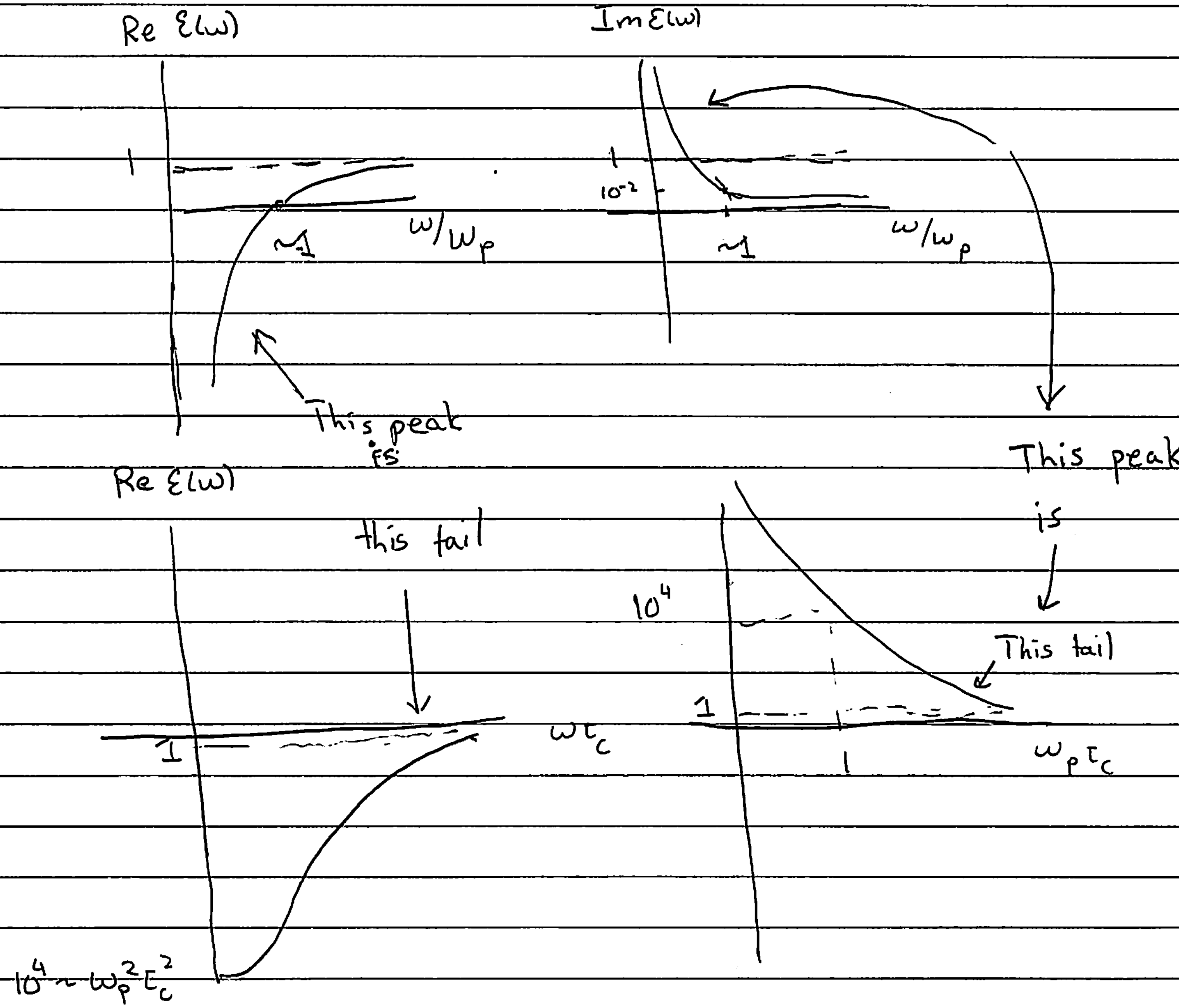
$$\mathrm{Re}\,\mathcal{E}(\omega) = 1 - \frac{\sigma_0\tau_c}{1+(\omega\tau_c)^2} = 1 - \frac{\omega_p^2\tau_c^2}{1+(\omega\tau_c)^2}$$

$$\mathrm{Im}\,\mathcal{E}(\omega) = \frac{\sigma_0/\omega}{(1+\omega^2\tau_c^2)} = \frac{\omega_p^2\tau_c/\omega}{(1+(\omega\tau_c)^2)}$$

Then for $\omega \longrightarrow \infty$

$$\mathrm{Re}\,\varepsilon(\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2}$$

$$\mathrm{Im}\,\varepsilon \simeq \frac{\omega_p^2}{\omega^3 \tau_c}$$



Excercise 3   Since

$$\partial_t \rho + \nabla \cdot j = 0$$

Using

$$j(t) = \int \sigma(t-t') E(t')$$

$$\nabla \cdot j(t) = \int \sigma(t-t') \nabla \cdot E(t')$$

$$= \int \sigma(t-t') \rho(t')$$

Thus

$$\partial_t \rho + \int \sigma(t-t') \rho(t') = 0$$

Excercise 4   The green fcn satifies

$$\left[m \frac{d}{d\tau} + \frac{m}{\tau_c} \right] G(t, t_0) = \delta(t - t_0)$$

So:

The solution to

$$m \frac{dV}{dt} + \frac{m}{\tau_c} V = e E(t_0)$$

is $V(t) = \int G(t-t_0) \, e E(t_0)$

So

$$J(t) = Nev(t) = \int G_R(t-t') \; Ne^2 E(t') \; dt$$

So

$$j(\omega) = \underbrace{G_R(\omega) \; Ne^2}_{\equiv \sigma(\omega)} \; E(\omega)$$

Thus we see that

$$\sigma(\omega) = Ne^2 G_R(\omega)$$

Excercises

To find the green fcn in time we integrate across the δ' fcn

$$G(t_0+\varepsilon, t_0) - \cancel{G(t_0-\varepsilon, t_0)} = \frac{1}{m} \int \delta(t-t_0) = \frac{1}{m}$$

The homogeneous solution is

$$G(t, t_0) = Ce^{-\tau/\tau_c} \qquad t > t_0$$

Together with the B.C. gives

$$G(t, t_0) = \frac{\Theta(\tau)}{m} e^{\tau/\tau_c} \qquad \tau \equiv t - t_0$$

So we see that:

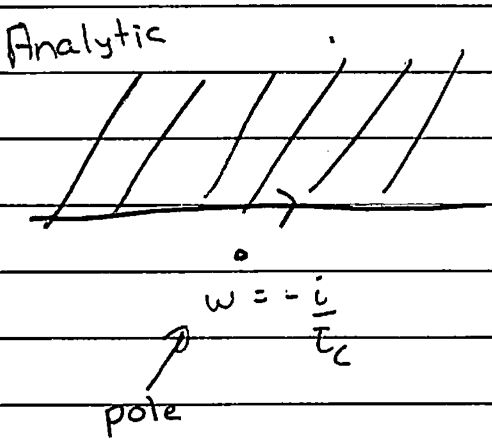
$$\partial_t \rho + \int_{-\infty}^{t} \omega_p^2 \, e^{-(t-t')/\tau_c} \rho(t')\, dt' = 0$$

Excercise 6

To find $G(\tau)$ using fourier transforms:

$$G(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\tau_c/m \; e^{-i\omega t}}{1 - i\omega\tau_c}$$

$$\sigma = \frac{\frac{Ne^2}{m}\tau_c}{1 - i\omega\tau_c} = Ne^2 G_R$$



For $\tau < 0$ we close the contour in the UHP. Since the function is analytic we pick up no poles and get zero.

For $\tau > 0$ we close in lower half plane getting the pole

$$G(\tau) = -2\pi i \; \mathrm{Res}_{\omega = -i/\tau_c}$$

$$= -2\pi i \frac{\tau_c}{m} \frac{1}{i\tau_c} \frac{e^{-\tau/\tau_c}}{2\pi} \overbrace{\mathrm{Res} \frac{1}{\frac{1}{i\tau_c} - \omega}}^{=-1}$$

So

$$G(\tau) = \theta(\tau) \frac{e^{-\tau/\tau_c}}{m} \quad \text{as before}$$