\( \nabla \cdot E = \rho (w) \)
\[
\nabla \times B = \frac{\partial \mathbf{j}(w)}{\partial t} + \frac{-i\omega}{c} \mathbf{E} 
\]
\( \nabla \cdot B = 0 \)
\[
\nabla \times E = \frac{i\omega}{c} \mathbf{B} 
\]

In order to specify the currents we postulated a constituent relation

\[ \mathbf{j}(w) = \sigma(w) \mathbf{E}(w) \equiv -i\omega \chi_e(w) \mathbf{E}(w) \]

Then we showed that the EOM are the same as with \( \mathbf{E}(w) = 1 + \chi_e(w) \). For wavelike solutions, \( \mathbf{E}(t, x) = \mathbf{E}_0 e^{i k x - (\omega t)} \) the Helmholtz equation, \( \left[ \frac{\nabla^2 + \omega^2 \mathbf{E}(w)}{c^2} \right] \mathbf{E} = 0 \), becomes:

\[
\left[ \frac{-k^2 + \omega^2 \mathbf{E}(w)}{c^2} \right] = 0 
\]

This defines the dispersion curve \( \omega(k) \).
So a general solution will be of the a superposition of waves of the form

\[ E_F e^{-i\omega(k)t} e^{ik \cdot x} \]

Where \( \omega(k) \) has real and imaginary parts

\[ E_F e^{-i \text{Re} \omega(k)t} e^{-\text{Im} \omega(k)t} e^{ik \cdot x} \quad (\text{Im} \omega(k) \text{ will be negative}) \]

For small imaginary part for \( \varepsilon = \varepsilon' + i\varepsilon'' \)

\[ \omega(k) \approx \frac{\varepsilon k (1 - i\varepsilon'')}{\sqrt{\varepsilon}} \quad n = \sqrt{\varepsilon} = n' + i n'' \]

So the imaginary part of \( \varepsilon \) records the absorption of the wave:

\[ \text{Re} n \]

\[ \text{Im} n \]
We had a simple model which described most of these features.

- Most of the essential features are a consequence of causality. Since $\chi(t)$ is an causal (vanishes for $t < 0$)

$$
\chi(\omega) = \int_{-\infty}^{\infty} e^{i\omega t} \chi(t) \, dt
$$

$\chi(\omega)$ is analytic in the UHP. As such the Kramers-Kronig relations apply. The Kramers-Kronig relations apply to any causal function, e.g. $\sigma(\omega), G_E(\omega), \ldots$ (provided the function decays sufficiently rapidly as $\omega \to \infty$)
Preliminaries for Kramers-Kronig

We talked about $\delta$-funs as a sequence of integrals

$$\int dx \, f(x) \, \delta(x) = \lim_{\varepsilon \to 0} \int dx \, f(x) \, \frac{\varepsilon}{\pi \, x^2 + \varepsilon^2} = f(0)$$

Similarly, principal value integrals are a limit of a sequence of funs:

$$\text{P} \int \frac{dx \, f(x)}{x-x_0} = \int_{-\infty}^{x_0-\varepsilon} \frac{f(x)}{x-x_0} + \int_{x_0+\varepsilon}^{+\infty} \frac{f(x)}{x-x_0}$$

one definition

$$= \lim_{\varepsilon \to 0} \int dx \, f(x) \, \frac{(x-x_0)}{(x-x_0)^2 + \varepsilon^2}$$

another

So we get the identity:

$$\frac{1}{x + i\varepsilon} = \frac{x - i \varepsilon}{x^2 + \varepsilon^2} = \text{P} \frac{1}{x} - i \text{Im} \delta(x)$$
Kramers-Kronig Relations

- What are they?

- For any function analytic in the upper half plane, i.e., the Fourier transform of a causal function such as $\sigma(w)$ and $G_p(w)$, $\chi_e(w)$. This is a relation between the real and imaginary parts.

\[
\text{Re} \, \chi_e(w_0) = \frac{2}{\pi} \text{P} \int_0^\infty \frac{\omega \, \text{Im} \chi_e(w)}{w^2 - w_0^2} \, dw
\]

\[
\text{Im} \, \chi_e(w_0) = -2w_0 \text{P} \int_0^\infty \frac{\text{Re} \chi(w)}{w^2 - w_0^2} \, dw
\]

Thus, if you know the real part of the dielectric constant, then you know the imaginary (absorptive) part of the dielectric constant and vice-versa.

Also written, since $\chi_e(-w) = \chi_e^*(w)$ (symmetric),

\[
\text{Re} \, \chi_e(w) = \frac{\text{P}}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} \chi_e(w)}{w - w_0} \, dw
\]

\[
\text{Im} \, \chi_e(w) = \frac{\text{P}}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} \chi_e(w)}{w - w_0} \, dw
\]
Uses of Kramers-Kronig

Thus, suppose Im $\varepsilon$ has a strong absorption peak at $\omega = \omega^*_e$:

\[ \text{Im} \varepsilon(\omega) = C \delta(\omega - \omega^*_e) \]

Then the real part near for $\omega$ near $\omega^*_e$ is:

\[ \text{Re} \varepsilon(\omega) \sim \frac{2}{\pi} \text{P} \int \frac{\omega \delta(\omega - \omega^*_e)}{\omega^2 - \omega_0^2} \, d\omega \]

\[ \sim \frac{2C}{\pi} \frac{\omega^*_e}{\omega^2 - \omega_0^2} \xrightarrow{\omega \rightarrow \omega^*_e} -\frac{C}{\pi} \frac{\omega^*_e}{\omega_0 - \omega^*_e} \]

Thus we can begin to see why the SHO model for the dielectric function gave reasonable qualitative agreement. Most of the coarse features of the dielectric function are dictated by causality (i.e. the Kramers-Kronig relations).
Kramers-Kronig Relations Proof

(and thus analytic)

* Since G is causal and there exist relations between the real and imaginary parts

Using Cauchy Thrm

\[ G(z) = \oint \frac{G(w)}{w-z} \, dw \]

Sending \( z = \omega_0 + i\varepsilon \)

\[ G_\varphi(\omega_0) = \int \frac{1}{2\pi i} \frac{G(w)}{w-\omega_0-i\varepsilon} \, dw \]

Skip in class

Using

\[ \frac{1}{\omega-\omega_0-i\varepsilon} = P \left( \frac{1}{\omega-\omega_0} \right) + i\pi \delta(\omega-\omega_0) \]

Finding

\[ G(\omega_0) = \left[ P \int_{-\infty}^{\infty} \frac{G(w)}{2\pi i} \, dw \right] \frac{1}{2} + \frac{1}{2} G(\omega_0) \]
Find

\[ \text{Re} G(\omega) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\text{Im} G_\omega(\omega)}{\omega - \omega_0} \, d\omega \]

\[ \text{Im} G(\omega) = -\frac{2}{\pi} \int_{-\infty}^{\infty} \frac{\text{Re} G_\omega(\omega)}{\omega - \omega_0} \, d\omega \]

Using the relation

\[ G^*(\omega) = \int_{0}^{\infty} e^{-i\omega t} G_\omega(t) = G_\omega(\omega) \]

One finds

\[ \text{Re} \chi(\omega) = \frac{2}{\pi} \int_{0}^{\infty} \frac{\omega \text{Im} \chi_e(\omega)}{\omega^2 - \omega_0^2} \, d\omega \]

\[ \text{Im} \chi(\omega) = -2 \frac{\text{Re} \chi_e(\omega)}{\pi} \int_{0}^{\infty} \frac{\text{Re} \chi_e(\omega)}{\omega^2 - \omega_0^2} \, d\omega \]

Thus, if you know the absorptive part of the dielectric constant, you can determine the real part of the dielectric constant and vice versa.
Optics of Metals

We described a metal with a classical transport (Drude Model)

- Electrons are free but experience a drag force

\[
m \frac{dV}{dt} + m \frac{V}{\tau_c} = eE(t)
\]

Found the d.c. conductivity number of electrons

\[
\sigma_0 = \omega_F^2 \tau_c \quad \omega_F \sim 10^{15} \text{ } 1/s \equiv \frac{n e^2}{m}
\]

\[
\approx 10^{18} \text{ } 1/s \quad \tau_c \sim 10^{13} \text{ seconds} = \text{time between collisions with impurities}
\]

Exercise 1. Show that the constituent relation in this model is

\[
\tilde{g}(\omega) = \cdot \sigma(\omega) E(\omega)
\]

is \[ \sigma(\omega) = \frac{\sigma_0}{1 - i \omega \tau_c} \]
Excercise 2
- Determine the low frequency limits of $\varepsilon(\omega)$ and $\sigma(\omega)$
- Sketch the real and imaginary parts of $\varepsilon(\omega)$ for a typical metal
- Describe what the high frequency behavior means for the propagation of light
- Such a Drake model gives a not too unrealistic picture of $\varepsilon(\omega)$

Excercise 3
- Show that charge relaxation is governed by the equation
  \[ \partial_t \rho + \int_0^\infty \sigma(t-t') \rho(t') \, dt' = 0 \]

Excercise 4
- Show that $\sigma(t-t')$ is determined up to a constant by the Green's function at the differential equation
  \[ \left[ \frac{m}{\epsilon_2} \frac{d}{dt} + \frac{m}{\epsilon_2} \right] G_R(t, t_0) = \delta(t-t_0) \]
Exercise 5

Determine $G_R(t,t')$ by direct integration in time, and infer $\sigma_1(t-t')$. What time scale controls the relaxation of charge, $\sigma_1$, $\nu_1$, $T_c$? Estimate numerically.

Exercise 6

By Fourier transforming $\sigma(w)$ determine $\sigma(t)$ where $t = t-t'$

Use complex variables and explain the differences between $T>0$ and $T<0$.
Solutions

Exercise 1. Fourier transforming the equation

\[-im \omega V(\omega) + \frac{m V(\omega)}{\tau_c} = \frac{e E(\omega)}{\tau_c}\]

\[j(\omega) = Ne V(\omega) = \frac{Ne^2/m}{-i\omega + \frac{1}{\tau_c}} E(\omega)\]

So \[j(\omega) = \left(\frac{Ne^2/m}{1 - i\omega \tau_c}\right) E(\omega)\]

\[= \frac{\sigma_0}{1 - i\omega \tau_c}\]

Exercise 2. For small frequency \(\omega\):

\[\varepsilon(\omega) = 1 + \varepsilon_0(\omega) \approx 1 + i\frac{\sigma_0}{\omega}\]

For general frequencies:

\[\varepsilon(\omega) = 1 + i\frac{\sigma_0}{\omega} \frac{1}{1 - i\omega \tau_c}\]

\[\text{Re}\varepsilon(\omega) = 1 - \frac{\sigma_0 \tau_c}{1 + \omega^2 \tau_c^2} = 1 - \frac{\omega_p^2 \tau_c^2}{1 + \omega^2 \tau_c^2}\]

\[\text{Im}\varepsilon(\omega) = \frac{\sigma_0 \omega}{(1 + \omega^2 \tau_c^2)} = \frac{\omega_p^2 \tau_c / \omega}{(1 + (\omega \tau_c)^2)}\]
Then for $\omega \to \infty$

$$\text{Re} \, \varepsilon(\omega) \approx 1 - \frac{\omega_p^2}{\omega^2}$$

$$\text{Im} \, \varepsilon \approx \frac{\omega_p^2}{\omega^3 \epsilon_c}$$

---

Diagram:

- **Re $\varepsilon(\omega)$**
  - As $\omega \to \omega_p$, $\varepsilon$ approaches a peak.
  - At $\omega = \omega_p$, $\varepsilon$ has a sharp rise.
  - After $\omega_p$, $\varepsilon$ decreases.

- **Im $\varepsilon(\omega)$**
  - As $\omega \to \omega_p$, $\varepsilon$ increases sharply.
  - After $\omega_p$, $\varepsilon$ decreases to zero.

- **Re $\varepsilon(\omega)$**
  - At $\omega = \omega_c$, $\varepsilon$ has a peak.
  - At $\omega = \omega_p$, $\varepsilon$ decreases sharply.

- **Im $\varepsilon(\omega)$**
  - As $\omega \to \omega_p$, $\varepsilon$ increases sharply.
  - After $\omega_p$, $\varepsilon$ decreases to a constant value.

- **Other Notes**
  - The diagram shows the behavior of $\varepsilon$ as a function of $\omega$.
  - Key points include $\omega_p$ and $\omega_c$.
  - The peaks and tails are marked with arrows and labels.
Exercise 3. Since
\[ \partial_t p + \nabla \cdot j = 0 \]

Using
\[ j(t) = \int \sigma(t-t') E(t') \]
\[ \nabla \cdot j(t) = \int \sigma(t-t') \nabla \cdot E(t') \]
\[ = \int \sigma(t-t') \rho(t') \]

Thus
\[ \partial_t p + \int \sigma(t-t') \rho(t') = 0 \]

Exercise 4. The Green's function satisfies
\[ \left[ m \frac{d}{dt} + \frac{m}{T_c} \right] G(t, t_0) = \delta(t - t_0) \]

So:

The solution to
\[ m \frac{d}{dt} \frac{V}{T_c} + m \frac{V}{T_c} = eE(t) \]

is
\[ V(t) = \int G(t-t_0) eE(t_0) \]
\[ J(t) = N \nu(t) = \int_0^t G(t-t') N e^2 E(t') \, dt \]

So

\[ j(\omega) = G_R(\omega) N e^2 E(\omega) \]

\[ = \sigma(\omega) \]

Thus we see that

\[ \sigma(\omega) = N e^2 G_R(\omega) \]

\[ \text{Exercise 5} \]

To find the Green function in time we integrate across the source

\[ G(t)+e, t_0) - G(t)-e, t_0) = \int_0^t \delta(t-t') \]

The homogeneous solution is

\[ G(t, t_0) = C e^{-t/\tau_c}, \quad t > t_0 \]

Together with the B.C. gives

\[ G(t, t_0) = \Theta(t) \frac{e^{-t/\tau_c}}{m}, \quad \xi = t-t_0 \]
So we see that:

\[ \partial_t p + \int_{-\infty}^{t} \omega^2 e^{-(t-t')/\tau_c} p(t') \, dt' = 0 \]

**Exercise 6**

To find \( G(t) \) using Fourier transforms:

\[
G(t) = \int_{-\infty}^{\infty} \frac{2\pi m}{1 - i\omega \tau_c} e^{-i\omega t} \, d\omega
\]

For \( \tau < 0 \) we close the contour in the UHP. Since the function is analytic we pick up no poles and get zero.

For \( \tau > 0 \) we close in lower half plane getting the pole

\[
G(t) = -2\pi i \operatorname{Res}_{\omega = -i/t_c} \frac{1}{1 - i\omega \tau_c} \frac{1}{m i t_c 2\pi} = -1
\]

\[
= -2\pi i \frac{\tau_c}{m i t_c 2\pi} \frac{1}{1 - \omega - i/t_c} \operatorname{Res}_{\omega = -i/t_c} \frac{1}{1 - i\omega \tau_c} = -1
\]
S_0

G(t) = \Theta(t) \frac{e^{-t/t_c}}{m} \quad \text{as before}